

# ALCOVE GEOMETRY AND A TRANSLATION PRINCIPLE FOR THE BRAUER ALGEBRA

ANTON COX, MAUD DE VISSCHER, AND PAUL MARTIN

**ABSTRACT.** There are similarities between algebraic Lie theory and a geometric description of the blocks of the Brauer algebra in characteristic zero. Motivated by this, we study the alcove geometry of a certain reflection group action. We provide analogues of translation functors for a tower of recollement, and use these to construct Morita equivalences between blocks containing weights in the same facet. Moreover, we show that the determination of decomposition numbers for the Brauer algebra in characteristic zero can be reduced to a study of the block containing the weight 0. We define parabolic Kazhdan-Lusztig polynomials for the Brauer algebra and show in certain low rank examples that they determine standard module decomposition numbers and filtrations.

## 1. INTRODUCTION AND MOTIVATION

The Brauer algebra  $B_n(\delta)$  was introduced in 1937 [Bra37]. It is an enlargement of the symmetric group  $\Sigma_n$ , designed to impose an orthosymplectic algebraic condition, depending on  $\delta$ , on a Schur-Weyl dual algebraic group over the complex field. However it may be defined over an arbitrary ring  $K$ , for any  $n \in \mathbb{N}$  and  $\delta \in K$ . It has integral representations (in the sense of [Ben91]) that pass to simple modules over suitable splitting fields (sometimes called cell modules in modern parlance), constructed by Brown [Bro55]. This raises the problem of determining simple decomposition matrices for these key modules (and hence for indecomposable projective modules) over other extensions  $k$  ( $k$  a field equipped with the property of  $K$ -algebra). This long-standing problem remains open. The aim of the present work is twofold: to develop some tools to solve this problem, by constructing a formal ‘weight space’ with a geometry and associated functors on the module categories, and to propose a possible combinatorial framework (at least over  $\mathbb{C}$ ) in which the answer might be couched.

The problem can be addressed in two parts: first working over  $\mathbb{C}$ , and then over fields of prime characteristic. (The latter can be anticipated to be significantly harder, as the representation theory of  $B_n(\delta)$  contains the representation theory of the symmetric group  $\Sigma_n$ .) A significant step towards an answer came with the determination in 2005 of the blocks of the algebra over  $\mathbb{C}$  (and in 2006 of a geometric linkage principle in any characteristic different from 2) [CDM05, CDM06]. These results were obtained by using functors that allow the algebras for all  $n$  to be treated together (as previously used in [MW98, CGM03]). (An alternative approach to the characteristic 0 result via characteristic  $p$  has recently been developed by Donkin and Tange [DT08].)

The key observation that underpins the various geometric considerations in this paper is that the cell modules of  $B_n(\delta)$  may be indexed by certain orbits of lattice points in the

Euclidean space  $\mathbb{E}^N$ . The orbits are those of a reflection group  $A$ , where  $A$  is the limit of the usual type- $A$  reflection group action on Euclidean  $N$ -space  $\mathbb{E}^N$ . The  $A$ -action is a parabolic in the limit type- $D$  reflection group action on  $\mathbb{E}^N$  (as in finite rank), and the orbits of the  $D$ -action on coset space  $\mathbb{E}^N/A$  describe the blocks of  $B_n(\delta)$ -mod over  $\mathbb{C}$ . For this reason we will work over  $\mathbb{C}$  in this paper. Using this parabolic/reflection group formulation we are able to:

- (1) determine a translation principle (Morita equivalences between certain blocks);
- (2) compute appropriate Brauer analogues of the parabolic Kazhdan-Lusztig polynomials that determine decomposition matrices in Lie (quantum group) theory;
- (3) use these to encode the structure of the algebra in many special cases (for example in low rank, with the obvious conjecture that this extends to all cases).

Our methodology, and the structure of the paper, can be summarised as follows. The well-established root system/Weyl group analysis of high-weight theory reduces many questions in Lie representation theory (of algebraic and quantum groups) to geometry and combinatorics [Deo87, Jan03, Soe97b], once the Weyl group and affine Weyl group action on weight space has been determined. Of course these Weyl groups are reflection groups, one a parabolic in the other [Hum90], facilitating, for example, an alcove geometric description of blocks.

Note the obvious analogy with the role of reflection groups described above. It was this which motivated our formulation of the results in [CDM06] (guided by success with a similar approach to other ‘diagram’ algebras [MW03]). In Lie theory the Euclidean space is finite and the reflection group is infinite by virtue of being affine; here it is by virtue of unbounded rank. Nonetheless, all the geometric and combinatorial machinery goes through unchanged. The development of this analogy in Section 3 lies at the heart of our methodology.

Arguably one of the most beautiful machines that exists for computing decomposition matrices in any setting is the method of (parabolic) Kazhdan-Lusztig polynomials in Lie theory [AJS94, Soe97b]. Not all of the assumptions of this set-up hold for the Brauer algebra, but in Section 8 we show how to bring the two theories close enough together that parabolic Kazhdan-Lusztig (pKL) polynomials suitable for the Brauer algebra may be computed.

In the group (or quantum group) case one has pKL polynomials associated to alcoves in the alcove geometry, determining (at least for the  $q$ -group over  $\mathbb{C}$ , and its Ringel dual Hecke algebra quotient [Erd94]) decomposition matrices in alcove blocks. In general there is more than one block intersecting an alcove, but there is also a translation principle [Jan03], which states that all these blocks are Morita equivalent, and hence do indeed have the same decomposition matrices. In our case the pKL method formally assigns the same decomposition matrix to every alcove block. One is therefore led to seek a form of translation principle.

The Brauer algebras  $B_n(\delta)$  as  $n$  varies form a tower via an idempotent construction. In [CMPX06] we gave a general axiom scheme for studying such a tower as a *tower of recollement*. The advantage of studying algebras in such a tower is the existence of four functors: induction, restriction, globalisation, and localisation, which relate the representation theories of the different algebras in a compatible manner.

In Section 4 we will show how towers of recollement, when combined with a suitable description of the blocks in the tower, give rise to analogues of translation functors and corresponding Morita equivalences. These functors are defined using induction or restriction

functors followed by projection onto a block, and are similar in spirit to  $\alpha$ -induction and  $\alpha$ -restriction functors for the symmetric group [Rob61].

We apply this translation theory to the Brauer algebras in Section 6, with the aim of proving that two blocks corresponding to weights in the same facet have the same representation theory (Corollary 6.8). However, in order to do this we will need some additional functors, generalisations of induction and restriction, which are introduced in Section 5. We will also see that when  $\delta < 0$  there are translation equivalences *between* certain facets, which raises interesting questions as to the true geometric structure underlying the representation theory.

We can also consider an analogue of translation ‘onto a wall’ in Lie theory for towers of recollement. Using this we show that the decomposition matrix for the Brauer algebra is determined by the decomposition matrix for the block containing the weight 0 (Theorem 6.14).

In Section 7 we consider various graphs associated to each block (or each facet), and show that they are in fact all isomorphic. For alcove graphs we can define associated Kazhdan-Lusztig polynomials; using the graph isomorphisms these polynomials can more generally be associated to any block graph. In the final section we show that when  $\delta = 1$  these polynomials correctly predict decomposition numbers and filtrations in the alcove case for standard modules in low rank examples.

## 2. A REVIEW OF BRAUER ALGEBRA REPRESENTATION THEORY

In this section we will very briefly summarise the basic representation theory of the Brauer algebras that will be needed in what follows. Details can be found in [CDM05, CDM06]. In this paper, we will restrict our attention to the case where the ground field is  $\mathbb{C}$ .

The Brauer algebra  $B_n(\delta)$  is a finite dimensional algebra with parameter  $\delta \in \mathbb{C}$ . When  $\delta \notin \mathbb{Z}$  this algebra is semisimple, so we will henceforth assume that  $\delta$  is an integer. *We will also assume that  $\delta \neq 0$ .*

It will be convenient to use the usual graphical presentation of Brauer algebras. An  $(n, m)$  *Brauer algebra diagram* will consist of a rectangular frame with  $n$  marked points on the northern edge and  $m$  on the southern edge called *nodes*. Each of these sets will be numbered from 1 to  $n$  (respectively  $m$ ) from left to right. Each node is joined to precisely one other by a line; lines connecting the northern and southern edge will be called *propagating lines* and the remainder (*northern or southern*) *arcs*.

Multiplication of two  $(n, n)$  diagrams is by concatenation, where any diagram obtained with a closed loop is set equal to  $\delta$  times the same diagram with the loop removed. Two diagrams are equivalent if they connect the same pairs of nodes. The algebra obtained by taking linear combinations of  $(n, n)$  diagrams is a realisation of  $B_n$ . Note that  $\mathbb{C}\Sigma_n$  is isomorphic to the subalgebra of  $B_n$  spanned by diagram with only propagating lines. Moreover,  $B_n$  is generated by this subalgebra together with the elements  $X_{i,j}$  with  $1 \leq i, j \leq n$  consisting of  $n - 2$  propagating lines and arcs joining  $i$  and  $j$  on the northern (respectively) southern edges.

The Brauer algebra can also be constructed via ‘iterated inflations’ of the symmetric group [KX01], and thus is a cellular algebra. If  $\delta \neq 0$ , then it is even quasihereditary. The

standard modules  $\Delta_n(\lambda)$  are parameterised by partitions of  $n, n-2, \dots, 1/0$  (where the final term depends on the parity of  $n$ ), and we will denote the set of such by  $\Lambda_n$ . If  $\delta \neq 0$  then the same set parameterises the simple modules.

We have the following explicit construction of standard modules. Consider a Brauer diagram with  $n$  northern nodes and  $n-2t$  southern nodes, and with no southern arcs. Such a diagram must have exactly  $t$  northern arcs. We will denote this diagram by  $X_{v,1,\sigma}$ , where  $v$  denotes the configuration of northern arcs, 1 represents the fixed southern boundary, and  $\sigma \in \Sigma_{n-2t}$  is the permutation obtained by setting  $\sigma(i) = j$  if the  $i$ th propagating northern node from the left is connected to the southern node labelled by  $j$ .

The elements  $v$  arising as above will be called *partial one-row diagrams*, and the set of such will be denoted by  $V_{n,t}$ . If a node  $i$  in  $w \in V_{n,t}$  is not part of a northern arc we say that it is *free*. The vector space spanned by the set of diagrams of the form  $X_{w,1,id}$  where  $w \in V_{n,t}$  will be denoted  $I_n^t$ . Note that  $\Sigma_m$  acts on  $I_n^t$  on the right by permuting the southern nodes.

Given  $\lambda$  a partition of  $m = n-2t$ , let  $S^\lambda$  denote the Specht module corresponding to  $\lambda$  for  $\Sigma_m$ . Then the standard module  $\Delta_n(\lambda)$  can be realised (see [DWH99, Section 2] or [CDM05, Section 2]) in the following manner. As a vector space we have

$$\Delta_n(\lambda) = I_n^t \otimes S^\lambda. \quad (1)$$

An element  $b$  of  $B_n$  acts on  $d \in I_n^t$  from the right by diagram multiplication. If the resulting product has fewer than  $m$  propagating lines then we define the action of  $b$  on  $d \otimes S^\lambda$  to be 0. Otherwise the product will result in a diagram with exactly  $m$  propagating lines, but these may now be permuted. We transfer this permutation (thought of as an element of  $\Sigma_m$ ) through the tensor product to act on  $S^\lambda$ .

For  $\delta \neq 0$  (or  $n > 2$ ) there is an idempotent  $e_n \in B_n(\delta)$  such that  $e_n B_n(\delta) e_n \cong B_{n-2}(\delta)$ , and so there are associated localisation and globalisation functors  $F_n : B_n(\delta)\text{-mod} \rightarrow B_{n-2}(\delta)\text{-mod}$  and  $G_n : B_n(\delta)\text{-mod} \rightarrow B_{n+2}(\delta)\text{-mod}$ . In this way we can regard  $B_n(\delta)\text{-mod}$  as a full subcategory of  $B_{n+2}(\delta)\text{-mod}$ , and hence  $\Lambda_n \subset \Lambda_{n+2}$ . We set  $\Lambda = \lim_{n \rightarrow \infty} (\Lambda_n \cup \Lambda_{n+1})$ , the set of all partitions. We will abuse terminology and say that two labels are in the same block when the associated standard modules are in the same block.

In order to describe the main results in [CDM05] we will need some additional terminology. Recall that for a partition  $\lambda$  (which we will identify with its Young diagram), the *content* of the box in row  $i$  and column  $j$  of the diagram is defined to be  $j - i$ . A pair of partitions  $\mu \subset \lambda$  is said to be  $\delta$ -balanced if the following conditions are satisfied:

- (1) the boxes in the skew partition can be paired so that the contents of each pair sum to  $1 - \delta$ ;
- (2) if the skew partition contains boxes labelled by  $1 - \frac{\delta}{2}, -\frac{\delta}{2}$  and there is only one such box in the bottom row then the number of pairs of such boxes is even.

We say that two general partitions  $\lambda$  and  $\mu$  are  $\delta$ -balanced if the pairs  $\lambda \cap \mu \subset \lambda$  and  $\lambda \cap \mu \subset \mu$  are both  $\delta$ -balanced. The importance of the  $\delta$ -balanced condition is clear from the following result [CDM05, Corollary 6.7]:

**Theorem 2.1.** *Two partitions  $\lambda$  and  $\mu$  are in the same block for  $B_n(\delta)$  if and only if they are  $\delta$ -balanced.*

Denote by  $V_\delta(\lambda)$  the set of partitions  $\mu$  such that  $\mu$  and  $\lambda$  are  $\delta$ -balanced. Note that if  $\mu \in V_\delta(\lambda)$  then so too are  $\lambda \cap \mu$  and  $\lambda \cup \mu$ . Thus  $V_\delta(\lambda)$  forms a lattice under the inclusion relation. We say that  $\mu$  is a *maximal balanced subpartition* of  $\lambda$  if  $\mu \in V_\delta(\lambda)$  and there does not exist  $\tau \in V_\delta(\lambda)$  with  $\mu \subset \tau \subset \lambda$ . One of the main steps in the proof of Theorem 2.1 is [CDM05, Theorem 6.5], which shows that if  $\mu$  is a maximal balanced subpartition of  $\lambda$  then

$$\text{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \neq 0.$$

The standard module  $\Delta_n(\lambda)$  has simple head  $L_n(\lambda)$  and all other composition factors are of the form  $L_n(\mu)$  where  $\mu \supset \lambda$  lies in the same block as  $\lambda$  [CDM05, Proposition 4.5]. If  $\lambda$  and  $\mu$  are such a pair with  $|\lambda/\mu| = 2$  then

$$[\Delta_n(\mu) : L_n(\lambda)] = \dim \text{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) = 1 \quad (2)$$

by [DWH99, Theorem 3.4 and the remarks after Theorem 3.1] (see [CDM05, Theorem 4.4]). If  $\mu \subset \lambda$  are two weights in the same block and  $\lambda/\mu = (a^b)$  for some  $a$  and  $b$  with  $a$  even then we also have [CDM05, Proposition 5.1] that

$$[\Delta_n(\mu) : L_n(\lambda)] = 1. \quad (3)$$

In general, if  $\lambda \vdash n$  and  $\mu \vdash m$  with  $m \leq n$  the exactness of the localisation functor implies that

$$[\Delta_N(\mu) : L_N(\lambda)] = [\Delta_n(\mu) : L_n(\lambda)]$$

for all  $N > n$ .

The algebra  $B_n(\delta)$  embeds inside  $B_{n+1}(\delta)$ , and so we may consider the associated induction and restriction functors  $\text{ind}_n$  and  $\text{res}_{n+1}$ . If  $\lambda$  and  $\mu$  are partitions we write  $\lambda \triangleright \mu$ , or  $\mu \triangleleft \lambda$  if the Young diagram for  $\mu$  is obtained from that for  $\lambda$  by removing one box. Then [DWH99, Theorem 4.1 and Corollary 6.4] we have short exact sequences

$$0 \rightarrow \bigoplus_{\mu \triangleleft \lambda} \Delta_{n+1}(\mu) \rightarrow \text{ind}_n \Delta_n(\lambda) \rightarrow \bigoplus_{\mu \triangleright \lambda} \Delta_{n+1}(\mu) \rightarrow 0 \quad (4)$$

and

$$0 \rightarrow \bigoplus_{\mu \triangleleft \lambda} \Delta_{n-1}(\mu) \rightarrow \text{res}_n \Delta_n(\lambda) \rightarrow \bigoplus_{\mu \triangleright \lambda} \Delta_{n-1}(\mu) \rightarrow 0. \quad (5)$$

The restriction rule for simples is not so straightforward. However we do have by [CDM05, Lemma 7.1] that if  $\mu$  is a partition obtained from  $\lambda$  by removing one box then

$$[\text{res}_n L_n(\lambda) : L_{n-1}(\mu)] \neq 0. \quad (6)$$

The next two results are new, and show how the local data in (4) and (5) can be applied to explicit decomposition number calculations (which illustrates one of the motivations for the tower of recollement formalism in [CMPX06]).

**Proposition 2.2.** *Suppose that by removing  $m$  boxes from  $\lambda \vdash n$  it is possible to reach a partition  $\mu \vdash n - m$  such that  $\Delta_{n-m}(\mu)$  is a projective  $B_{n-m}(\delta)$ -module. Then the simple module  $L_n(\lambda)$  does not appear as a composition factor in any  $\Delta_n(\nu)$  with  $\nu$  of degree less than  $n - 2m$ .*

*Proof.* Suppose that  $L_n(\lambda)$  does occur as a composition factor of  $\Delta_n(\nu)$ . Then by Brauer-Humphreys reciprocity [Don98, Proposition A2.2(iv)] the projective cover  $P_n(\lambda)$  of  $L_n(\lambda)$  has a standard module filtration with  $\Delta_n(\nu)$  as a factor.

As  $\Delta_{n-m}(\mu)$  is projective, so is  $\text{ind}_{n-1} \cdots \text{ind}_{n-m} \Delta_{n-m}(\mu)$ . By repeated application of (4) we see that this contains  $L_n(\lambda)$  in its head, and so must have as a summand  $P_n(\lambda)$ . We have that

$$\text{ind}_{B_{n-m}}^{B_n} = \text{res}_{B_n}^{B_{m+n}} G_{n+m-2} \cdots G_{n-m+2} G_{n-m}$$

and

$$G_r(\Delta_r(\lambda)) \cong \Delta_{r+2}(\lambda).$$

Therefore by repeated application of (5) to  $\Delta_{n+m}(\mu)$  we see that  $P_n(\lambda)$  cannot have a standard module filtration with  $\Delta_n(\nu)$  as a factor. This gives the desired contradiction and so we are done.  $\square$

**Remark 2.3.** Note that any standard module which is alone in its block must be projective. Thus there are many circumstances where Proposition 2.2 will be easy to apply. Indeed, this case will be sufficient for our purposes.

If  $\mu \subset \lambda$  are two partitions then their skew  $\lambda/\mu$  can be regarded as a series of disjoint partitions; when considering such differences we will list the various partitions in order from top right to bottom left. Thus a skew partition  $((22)^2)$  will consist of two disjoint partitions of the form  $(22)$ .

**Proposition 2.4.** *If  $\mu \subset \lambda$  is a balanced pair with  $\lambda/\mu = ((22)^2)$  or  $\lambda/\mu = ((1)^4)$  then*

$$\text{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \neq 0.$$

*Proof.* We may assume that  $\lambda \vdash n$  by localisation. If  $\lambda/\mu = ((22)^2)$  let  $\lambda'$  be  $\lambda$  less one of the two removable boxes in  $\lambda/\mu$ , and  $\mu'$  be the partition  $\mu$  together with the addable box from the other component of the skew. If  $\lambda/\mu = ((1)^4)$  then let  $\lambda'$  be  $\lambda$  with any one of the boxes in  $\lambda/\mu$  removed, and  $\mu'$  be  $\mu$  together with the unique box in  $\lambda/\mu$  making this a balanced pair.

In each case  $\mu'$  is a maximal balanced subpartition of  $\lambda'$  and so by [CDM05, Theorem 6.5] we have that

$$\text{Hom}_{n-1}(\Delta_{n-1}(\lambda'), \Delta_{n-1}(\mu')) \neq 0.$$

By [CDM05, Corollary 6.7] and (5) the only term in the block labelled by  $\lambda'$  in the standard filtration of  $\text{res}_n \Delta_n(\mu)$  is  $\Delta_n(\mu')$ . Therefore by Frobenius reciprocity we have

$$\text{Hom}(\text{ind}_{n-1} \Delta_{n-1}(\lambda'), \Delta_n(\mu)) \cong \text{Hom}(\Delta_{n-1}(\lambda'), \text{res}_n \Delta_n(\mu)) \neq 0. \quad (7)$$

Now  $\lambda$  is not the only weight in its block in the set of weights labelling term in the standard filtration of  $\text{ind}_{n-1} \Delta_{n-1}(\lambda')$ . However, by [CDM05, Lemma 4.10] it follows from (7) that

$$\text{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \neq 0$$

as required.  $\square$

In [CDM06] we identified partitions labelling Brauer algebra modules with elements of  $\mathbb{Z}^N$  (for suitable  $N$ ) only after transposition of the original partition to form its conjugate. Henceforth when we regard  $\Lambda_n$  (or  $\Lambda$ ) as a subset of  $\mathbb{Z}^\infty$  it will always be via this *transpose map*  $\lambda \rightarrow \lambda^T$ .

## 3. BRAUER ANALOGUES OF WEYL AND AFFINE WEYL GROUPS

We wish to identify reflection groups associated to the Brauer algebra which play the role of the Weyl and affine Weyl groups for reductive algebraic groups. First let us recall the properties of Weyl groups which we wish to replicate.

In Lie theory a Weyl group  $W$  is a reflection group acting on a Euclidean weight space with the following properties:

- (1) There is an integral set of weights on which  $W$  acts via a ‘dot’ action  $(W, \cdot)$ ,
- (2) The reflection hyperplanes of  $W$  under this action break space up into chambers (and other facets),
- (3) A complete set of weights indexing simple (or standard) modules coincides with the weights in a single chamber under the dot action, namely that containing the zero weight. Such weights are said to be *dominant*.

Thus the selection of an indexing set for the dominant weights is taken care of by the Weyl group (and its dot action). In positive characteristic  $p$  or at a quantum  $l$ th root of unity there is then a second stage, the introduction of an affine extension of  $W$  (with action depending on  $p$  or  $l$ ), which has orbits whose intersection with the dominant weights determine the blocks.

This affine extension defines an additional set of reflecting hyperplanes, which break the set of weights up into a series of chambers (now called alcoves) and other facets. We refer to this configuration of facets, together with the action of the affine extension, as the *alcove geometry* associated to the particular Lie theory in question.

The alcove geometry controls much of the representation theory of the corresponding reductive group. In particular, we typically have a *translation principle* which says that there are Morita equivalences between blocks which intersect a given facet, and so much of the representation theory does not depend on the weight itself but only on the facet in which it lies.

We will show how a version of the above programme can be implemented for the Brauer algebra from scratch.

Let  $\mathbb{E}^n$  be the  $\mathbb{R}$ -vector space with basis  $e_1, \dots, e_n$ , and  $\underline{n} = \{1, \dots, n\}$ . We will define various reflections on  $\mathbb{E}^n$  corresponding to the standard action of the type  $D$  Weyl group. Let  $(ij)$  be the reflection in the hyperplane in  $\mathbb{E}^n$  through the origin which takes  $e_i$  to  $e_j$  and fixes all other unit vectors, and  $(ij)_-$  to be the reflection in the hyperplane perpendicular to  $e_i + e_j$  which takes  $e_i$  to  $-e_j$ . We define

$$W_a(n) = \langle (i, j), (i, j)_- : i \neq j \in \underline{n} \rangle$$

which is the type  $D$  Weyl group. Note that it has a subgroup

$$W(n) = \langle (i, j) : i \neq j \in \underline{n} \rangle$$

which is just the type  $A$  Weyl group (isomorphic to  $\Sigma_n$ ).

As explained in the previous section, the Brauer algebras  $B_n(\delta)$  as  $n$  varies form a tower. Thus it is natural to consider all such algebras simultaneously. In order to do this we will work with the infinite rank case. Note also that orbits of the finite Weyl group  $W(n)$  are

not sufficient to define an indexing set for the simple  $B_n(\delta)$ -modules (one needs to consider  $W(n+1)$ -orbits, but then this group is not a subgroup of  $W_a(n)$ ), unlike the infinite rank case.

Let  $\mathbb{E}^\infty$  be the  $\mathbb{R}$ -vector space consisting of (possibly infinite) linear combinations of the elements  $e_1, e_2, \dots$ . We say that  $\lambda \in \mathbb{Z}^\infty$  has *finite support* if only finitely many components of  $\lambda$  are non-zero, and write  $\mathbb{Z}^f$  for the set of such elements. (We define  $\mathbb{E}^f$  similarly.) Thus, with the obvious embedding of  $\mathbb{Z}^n$  inside  $\mathbb{Z}^{n+1}$ , we have that  $\mathbb{Z}^f = \lim_{n \rightarrow \infty} \mathbb{Z}^n$ . (Note that the usual component-wise inner product on  $\mathbb{Z}^\infty$  is finite on  $\mathbb{Z}^f$ , so that  $\mathbb{Z}^f$  is just the integral part of the usual  $l^2$  Hilbert space in  $\mathbb{R}^\infty$ .)

We say that an element  $\lambda = (\lambda_1, \lambda_2, \dots)$  of  $\mathbb{Z}^f$  is *dominant* if  $\lambda_i \geq \lambda_{i+1}$  for all  $i$ . (Note that any such element must lie in  $\mathbb{N}^\infty$ .) Embed  $\mathbb{E}^n$  inside  $\mathbb{E}^\infty$  in the obvious way, and let  $W_a$ , and  $W$  be the corresponding limits of  $W_a(n)$ , and  $W(n)$ . Clearly the space  $\mathbb{Z}^f$  is closed under the action of  $W_a$ . We will call elements in  $\mathbb{Z}^f$  *weights*. Dominant weights are precisely those which label standard modules for the Brauer algebras, and by analogy with Lie theory we will denote the set of such weights by  $X^+$ . General elements of  $\mathbb{E}^\infty$  will be called *vectors*. We will use Greek letters for weights and Roman letters for general vectors.

Given a reflection group  $G$  (or the corresponding set of hyperplanes  $\mathbb{H}$ ), we say that a vector is *regular in  $G$*  (or in  $\mathbb{H}$ ) if it lies in the interior of a chamber, i.e. in some connected component of  $\mathbb{E}^\infty \setminus \bigcup_{X \in \mathbb{H}} X$ . Otherwise we say the vector is *singular*. In the case  $G = W_a$  we shall call chambers *alcoves* to emphasise the distinction between this and the  $W$  case. For  $v \in \mathbb{E}^\infty$  we define the *degree of singularity*

$$s(v) = |\{\{i, j\} : v_i = \pm v_j, \quad i \neq j\}|$$

(which need not be finite in general). Note that a vector  $v$  is regular in  $W_a$  if and only if  $s(v) = 0$ . The next lemma is clear.

**Lemma 3.1.** (i) *There is a chamber  $A^+$  of the action of  $W$  on  $\mathbb{E}^\infty$  consisting of all strictly decreasing sequences.*

(ii) *The boundary of  $A^+$  consists of all non-strictly decreasing sequences.*

Recall that in Lie theory we typically consider a shifted reflection group action with respect to some fixed element  $\rho$ . It will be convenient to consider a similar adjustment here. Let  $-2\omega = (1, 1, \dots, 1) \in \mathbb{E}^\infty$  and  $\rho_0 = (0, -1, -2, \dots)$ . For  $\delta \in \mathbb{Z}$  define

$$\rho_\delta = \rho_0 + \delta\omega.$$

For  $w \in W_a$  and  $v \in \mathbb{E}^\infty$  let

$$w \cdot_\delta v = w(v + \rho_\delta) - \rho_\delta$$

where the right-hand side is given by the usual reflection action of  $W_a$  on  $\mathbb{E}^\infty$ . Note that  $\mathbb{Z}^f$  is closed under this action of  $W_a$ . We say that a weight  $\lambda$  is  $\delta$ -*regular* if  $\lambda + \rho_\delta$  is regular, and define the *degree of  $\delta$ -singularity* of  $\lambda$  to be  $s(\lambda + \rho_\delta)$ .

**Proposition 3.2.** *Let  $\lambda \in \mathbb{Z}^f$ .*

(i) *For  $w \in W$  the weight  $w \cdot_\delta \lambda$  does not depend on  $\delta$ . Moreover, if  $w \neq 1$  and  $\lambda \in X^+$  then  $w \cdot_\delta \lambda \notin X^+$ .*

(ii) *If  $\lambda \in X^+$  then  $\lambda + \rho_\delta$  can only lie on a  $(ij)_-$ -hyperplane.*

(iii) *We have  $\lambda \in X^+$  if and only if  $\lambda + \rho_\delta \in A^+$ .*

*Proof.* (i) Note that

$$(ij)(\lambda + \rho_0 + \delta\omega) - \rho_0 - \delta\omega = (ij)(\lambda + \rho_0) - \rho_0.$$

(ii) and (iii) are clear.  $\square$

The description of the blocks of the Brauer algebra in characteristic zero in Theorem 2.1 was given the following geometric reformulation in [CDM06]:

**Theorem 3.3.** *Two standard modules  $\Delta_n(\lambda)$  and  $\Delta_n(\lambda')$  for  $B_n(\delta)$  are in the same block if and only if  $\lambda^T$  and  $\lambda'^T$  are in the same  $(W_a(n), \cdot_\delta)$ -orbit.*

**Remark 3.4.** In summary, we have shown that there is a space that plays a role analogous to a weight space (in Lie theory) for the Brauer algebra, together with an action of the type  $A$  Coxeter group which plays the role of the Weyl group, while the corresponding type  $D$  Coxeter group plays the role of the affine Weyl group.

We will now consider the geometry of facets induced by  $W_a$  inside  $A^+$ . We will call reflection hyperplanes *walls*, and for any collection of hyperplanes we will call a connected component of the set of points lying on the intersection of these hyperplanes but on no other a *facet*. (Then an alcove is a facet corresponding to the empty collection of hyperplanes.)

It will be convenient to have an explicit description of the set of vectors in a given facet. For vectors in  $A^+$  (which will be the only ones which concern us) these facets are defined by the hyperplanes  $v_i = -v_j$  for some  $i \neq j$ . For  $v = (v_1, v_2, v_3, \dots)$  in  $A^+$ , note that for all  $i \in \mathbb{N}$  we have

$$|\{j : |v_j| = |v_i|\}| \leq 2.$$

We will call  $v_i$  a *singleton* if  $v_i$  is the only coordinate with modulus  $|v_i|$ , and the pair  $v_i, v_j$  a *doubleton* if  $|v_i| = |v_j|$  and  $i \neq j$ .

For a given facet  $F$  with  $v \in F$ , a vector  $v' \in A^+$  lies in  $F$  if and only if  $|v'_i| = |v'_j|$  whenever  $|v_i| = |v_j|$ , and  $|v'_i| > |v'_j|$  whenever  $|v_i| > |v_j|$ . Therefore an alcove (where every  $v_i$  is a singleton) is determined by a permutation  $\pi$  from  $\mathbb{N}$  to  $\mathbb{N}$  where  $|v_{\pi(n)}|$  is the  $n$ th smallest modulus occurring in  $v$ . Note that not every permutation corresponds to an alcove in this way. Further, if  $i < \pi(1)$  then  $v_i > 0$ , while if  $i > \pi(1)$  then  $v_i < 0$ .

For more general facets we replace the permutation  $\pi$  by a function  $f : \mathbb{N} \rightarrow \mathbb{N} \cup (\mathbb{N} \times \mathbb{N})$  such that  $f(n)$  is the coordinate, or pair of coordinates, where the  $n$ th smallest modulus in  $v$  occurs. For example, if

$$v = (6, 4, 2, 1, 0, -2, -3, -5, \dots)$$

then the facet containing  $v$  corresponds to a function whose first four values are  $f(1) = 5$ ,  $f(2) = 4$ ,  $f(3) = (3, 6)$ , and  $f(4) = 7$ .

We will denote by  $A_0$  the alcove corresponding to the identity permutation. Thus  $A_0$  consists of all  $v \in A^+$  such that  $|v_1| < |v_2|$  and  $v_2 < 0$ . It is easy to see that, for any  $\delta \geq 0$ , the weight 0 is  $\delta$ -regular, with the vector  $0 + \rho_\delta$  in  $A_0$ ; in this case we will call the  $(W_a, \cdot_\delta)$ -alcove the  $\delta$ -fundamental alcove.

**Lemma 3.5.** *For  $\delta \geq 0$  the set of weights in the  $\delta$ -fundamental alcove is*

$$\{\lambda \in X^+ : \lambda_1 + \lambda_2 \leq \delta\}.$$

*Proof.* By our discussion above, the desired set of weights is precisely the set of dominant  $\lambda$  such that  $x = \lambda + \rho_\delta$  and  $|x_1| < |x_2|$ . But this means that

$$\lambda_1 - \frac{\delta}{2} < \frac{\delta}{2} + 1 - \lambda_2$$

which implies the result.  $\square$

**Remark 3.6.** Although our alcove geometry is reminiscent of that arising in positive characteristic Lie theory, there are also some striking differences. Consider for example the case when  $\delta = 1$ . The alcove  $A_0$  is non-empty and contains the two weights 0 and (1). The next lowest alcove contains (2, 1) and (2, 2), and the third contains (3, 2, 1) and (3, 1, 1). However, the associated facets with singularity 1 are not necessarily finite (in  $\mathbb{E}^\infty$ ); for example there is a facet consisting of the weights (2) and the  $n$ -tuple  $(1, \dots, 1)$  for all  $n \geq 2$ . In particular, not every weight on a wall is adjacent to a weight in an alcove.

Recall that for  $\delta \in \mathbb{N}$  the Brauer algebra  $B_r(\delta)$  is in Schur-Weyl duality with  $O_\delta(\mathbb{C})$  acting on the  $r$ th tensor product of the natural representation.

**Theorem 3.7.** *Suppose that  $\delta \in \mathbb{N}$ . The elements of  $\Lambda_\infty$  corresponding to weights in the  $\delta$ -fundamental alcove are in bijection with the set of partitions labelling the irreducible representations which arise in a decomposition of tensor powers of the natural representation of  $O_n(\mathbb{C})$ .*

*Proof.* For  $O_n(\mathbb{C})$  tensor space components are labelled by partitions whose first and second columns sum to at most  $n$  (see for example [GW98, Theorem 10.2.5]). The result now follows by comparing with Lemma 3.5 via the transpose map on partitions.  $\square$

**Remark 3.8.** The above result shows that the fundamental alcove arises naturally in the representation theory of  $O_n(\mathbb{C})$ .

Suppose that  $\delta < 0$ . Choose  $m \in \mathbb{N}$  so that  $\delta = -2m$  (if  $\delta$  is even) or  $\delta = -2m + 1$  (if  $\delta$  is odd). It is easy to see that 0 is  $\delta$ -singular of degree  $m$ . Indeed, any dominant weight  $\lambda$  is  $\delta$ -singular of degree at least  $m$ . Thus there are no regular dominant weights for  $\delta < 0$ . Instead of the  $\delta$ -fundamental alcove, we can consider the  $\delta$ -fundamental facet containing 0, for which we have

**Lemma 3.9.** *For  $\delta < 0$  of the form  $-2m$  or  $-2m + 1$  the set of weights in the  $\delta$ -fundamental facet is  $\{0\}$ .*

*Proof.* We consider the case  $\delta = -2m$ ; the odd case is similar. The element  $0 + \rho_\delta$  equals

$$(m, m - 1, \dots, 0, -1, \dots)$$

and hence our facet consists of all vectors of the form

$$(t, t - 1, \dots, -t + 1, -t, v_{|\delta|}, v_{|\delta|+1}, \dots)$$

where the sequence  $-t, v_{|\delta|} \dots$  is decreasing (as any other weight would be non-dominant). But this implies that  $Y_\delta = \{0\}$ .  $\square$

This is very different from the case  $\delta > 0$ . However we do have

**Theorem 3.10.** *Suppose that  $\delta = -2m$ . The set of elements of  $\Lambda_\infty$  corresponding to weights which can be obtained from 0 via a sequence of one box additions only involving intermediate weights of singularity  $m$  is in bijection with the set of partitions labelling the irreducible representations which arise in a decomposition of tensor powers of the natural representation of  $\mathrm{Sp}_{2m}(\mathbb{C})$ .*

*Proof.* First note that we can clearly add boxes in the first  $m$  coordinate of 0 without changing the degree of singularity. In order to change the  $(m+1)$ st coordinate in our path, we will have to pass through some point of the form

$$(a_1, a_2, \dots, a_m, 1, -1, -2, \dots).$$

where  $a_1 > a_2 > \dots > a_m > 1$ . But this implies that the first  $m+1$  coordinates of the vector all pair up with the corresponding negative values later down the vector, and so this is a singular vector of degree  $m+1$ . The result now follows from the description of tensor space components (see for example [GW98, Theorem 10.2.5]).  $\square$

We will see in Section 6 that there is a sense in which the set of weights occurring in Theorem 3.10 can be regarded as playing the role of an alcove in the  $\delta < 0$  case.

#### 4. A TRANSLATION PRINCIPLE FOR TOWERS OF RECOLLEMENT

Towers of recollement were introduced in [CMPX06] as an axiom scheme for studying various families of algebras. The Brauer algebra over  $\mathbb{C}$  was shown to satisfy these axioms in [CDM05]. We will prove a general result about Morita equivalences in such towers, and apply it in the following section to the Brauer algebra. In this section we will work over a general field  $k$ .

Suppose that we have a family of  $k$ -algebras  $A = \{A_n : n \in \mathbb{N}\}$  forming a tower of recollement. The precise properties of such a tower will not concern us here; such properties as we need will be introduced in what follows, and details can be found in [CMPX06].

In each  $A_n$  with  $n \geq 2$  there exists an idempotent  $e_n$  such that  $e_n A_n e_n \cong A_{n-2}$ . This determines a pair of functors: *localisation*  $F_n$  from  $A_n$ -mod to  $A_{n-2}$ -mod, and *globalisation*  $G_n$  from  $A_n$ -mod to  $A_{n+2}$ -mod given on objects by

$$F_n M = e_n M \quad \text{and} \quad G_n M = A_{n+2} e_{n+2} \otimes_{A_n} M.$$

The functor  $F_n$  is exact,  $G_n$  is right exact, and  $G_n$  is left adjoint to  $F_{n+2}$ . We also have algebra inclusions  $A_n \subset A_{n+1}$  for each  $n \geq 0$ , and associated functors  $\mathrm{ind}_n$  from  $A_n$ -mod to  $A_{n+1}$ -mod and  $\mathrm{res}_n$  from  $A_n$ -mod to  $A_{n-1}$ -mod. Let  $\Lambda_n$  be an indexing set for the simple  $A_n$ -modules. Globalisation induces an embedding of  $\Lambda_n$  inside  $\Lambda_{n+2}$ , and we take  $\Lambda$  to be the disjoint union of  $\lim_n \Lambda_{2n}$  and  $\lim_n \Lambda_{2n+1}$ , whose elements we call *weights*.

The algebras  $A_n$  are quasihereditary, and so there exists a standard  $A_n$ -module  $\Delta_n(\lambda)$  for each weight  $\lambda$  in  $\Lambda_n$ , such that the associated simple  $L_n(\lambda)$  arises as its head. The restriction of a standard module has a filtration by standards and we denote by  $\mathrm{supp}_n(\lambda)$  the multi-set of labels for standard modules occurring in such a filtration of  $\mathrm{res}_n \Delta_n(\lambda)$ . The embedding of  $\Lambda_n$  in  $\Lambda_{n+2}$  induces an embedding of  $\mathrm{supp}_n(\lambda)$  inside  $\mathrm{supp}_{n+2}(\lambda)$ , which becomes an identification if  $\lambda \in \Lambda_{n-2}$ . We denote by  $\mathrm{supp}(\lambda)$  the set  $\mathrm{supp}_n(\lambda)$  with  $n \gg 0$ .

Suppose that we have determined the blocks of such a family of algebras (or at least a linkage principle); we are thinking of the cases where we have an alcove geometry at hand, but will avoid stating the result in that form. Let  $\text{res}_n^\lambda$  be the functor  $\text{pr}_{n-1}^\lambda \text{res}_n$  and  $\text{ind}_n^\lambda$  be the functor  $\text{pr}_{n+1}^\lambda \text{ind}_n$  where  $\text{pr}_n^\lambda$  is projection onto the block containing  $\lambda$  for  $A_n$ . Note that  $\text{res}_n^\lambda$  is exact and  $\text{ind}_n^\lambda$  is right exact. We will regard these functors as analogues of *translation functors* in Lie theory.

Let  $\mathcal{B}_n(\lambda)$  denote the set of weights in the block of  $A_n$  which contains  $\lambda$ . Our embedding of  $\Lambda_n$  into  $\Lambda_{n+2}$  induces an embedding of  $\mathcal{B}_n(\lambda)$  into  $\mathcal{B}_{n+2}(\lambda)$ , and we denote by  $\mathcal{B}(\lambda)$  the corresponding limiting set. We will say that two elements  $\lambda$  and  $\lambda'$  are *translation equivalent* if they satisfy

- (i) The weight  $\lambda'$  is the only element of  $\mathcal{B}(\lambda') \cap \text{supp}(\lambda)$ .
- (ii) The weight  $\lambda$  is the only element of  $\mathcal{B}(\lambda) \cap \text{supp}(\lambda')$ .
- (iii) For all weights  $\mu \in \mathcal{B}(\lambda)$  there is a unique element  $\mu' \in \mathcal{B}(\lambda') \cap \text{supp}(\mu)$ , and  $\mu$  is the unique element in  $\mathcal{B}(\lambda) \cap \text{supp}(\mu')$ .

Clearly conditions (i) and (ii) are special cases of (iii); we list them separately as in an alcove geometry (i) and (ii) will be enough for (iii) to hold. When  $\lambda$  and  $\lambda'$  are translation equivalent then we will denote by  $\theta : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda')$  the bijection taking  $\mu$  to  $\mu'$ . We will see that translation equivalent weights belong to Morita equivalent blocks.

We will put a very crude partial order on weights in  $\mathcal{B}(\lambda)$  by saying that  $\lambda > \mu$  if there exists  $n$  such that  $\mu \in \Lambda_n$  and  $\lambda \in \Lambda_{n+2t}$  for some  $t \in \mathbb{N}$ , but  $\lambda \notin \Lambda_n$ . Note that this is the opposite of the standard order arising from the quasi-hereditary structure; we prefer to work with the natural order on the size of partitions. In the following proposition, by a unique element in a multi-set we mean one with multiplicity one.

**Proposition 4.1.** *Let  $A$  be a tower of recollement. Suppose that  $\lambda \in \Lambda_n$  and  $\lambda' \in \Lambda_{n-1}$  are translation equivalent, and that  $\mu \in \mathcal{B}_n(\lambda)$  is such that the  $\mu'$  is in  $\mathcal{B}_{n-1}(\lambda')$ . Then we have that*

$$\text{res}_n^{\lambda'} L_n(\mu) \cong L_{n-1}(\mu') \quad \text{and} \quad \text{ind}_{n-1}^\lambda L_{n-1}(\mu') \cong L_n(\mu) \quad (8)$$

for all  $\mu \in \mathcal{B}_n(\lambda)$ . Further if  $\tau \in \mathcal{B}_n(\lambda)$  is such that  $\tau'$  is in  $\mathcal{B}_{n-1}(\lambda')$  then we have

$$[\Delta_n(\mu) : L_n(\tau)] = [\Delta_{n-1}(\mu') : L_{n-1}(\tau')] \quad (9)$$

and

$$\text{Hom}(\Delta_n(\mu), \Delta_n(\tau)) \cong \text{Hom}(\Delta_{n-1}(\mu'), \Delta_{n-1}(\tau')). \quad (10)$$

*Proof.* We begin with (8). Consider the exact sequence

$$\Delta_n(\mu) \longrightarrow L_n(\mu) \longrightarrow 0.$$

Applying  $\text{res}_n^{\lambda'}$  we obtain by our assumptions the exact sequence

$$\Delta_{n-1}(\mu') \longrightarrow \text{res}_n^{\lambda'} L_n(\mu) \longrightarrow 0$$

and hence  $\text{res}_n^{\lambda'} L_n(\mu)$  has simple head  $L_{n-1}(\mu')$ , and possibly other composition factors  $L_{n-1}(\tau')$  with  $\tau' > \mu'$ .

If  $L(\tau')$  is in the socle of  $\text{res}_n^{\lambda'} L_n(\mu)$  then we have

$$\begin{aligned} \text{Hom}(\Delta_n(\tau), L_n(\mu)) &= \text{Hom}(\text{ind}_{n-1}^\lambda \Delta_{n-1}(\tau'), L_n(\mu)) \cong \text{Hom}(\text{ind}_{n-1} \Delta_{n-1}(\tau'), L_n(\mu)) \\ &\cong \text{Hom}(\Delta_{n-1}(\tau'), \text{res}_n L_n(\mu)) \cong \text{Hom}(\Delta_{n-1}(\tau'), \text{res}_n^{\lambda'} L_n(\mu)) \neq 0 \end{aligned}$$

by our assumptions and Frobenius reciprocity, which implies that  $\tau = \mu$  and hence  $\tau' = \mu'$ . Therefore  $\text{res}_n^{\lambda'} L_n(\mu) \cong L_{n-1}(\mu')$  as required.

Next consider the exact sequence

$$\Delta_{n-1}(\mu') \longrightarrow L_{n-1}(\mu') \longrightarrow 0.$$

Applying  $\text{ind}_{n-1}^\lambda$  we obtain by our assumptions the exact sequence

$$\Delta_n(\mu) \longrightarrow \text{ind}_{n-1}^\lambda L_{n-1}(\mu') \longrightarrow 0.$$

and hence  $\text{ind}_{n-1}^\lambda L_{n-1}(\mu')$  has simple head  $L_n(\mu)$ , and possibly other composition factors  $L_n(\tau)$  with  $\tau > \mu$ . Now apply  $\text{res}_n^{\lambda'}$  to obtain the exact sequence

$$\Delta_{n-1}(\mu') \longrightarrow \text{res}_n^{\lambda'} \text{ind}_{n-1}^\lambda L_{n-1}(\mu') \longrightarrow 0.$$

Then  $\text{res}_n^{\lambda'} \text{ind}_{n-1}^\lambda L_{n-1}(\mu')$  has simple head  $L_{n-1}(\mu')$  and possibly other composition factors  $L_{n-1}(\tau')$  with  $\tau' > \mu'$  corresponding to those in  $\text{ind}_{n-1}^\lambda L_{n-1}(\mu')$ . We have

$$\text{Hom}(L_{n-1}(\mu'), \text{res}_n^{\lambda'} \text{ind}_{n-1}^\lambda L_{n-1}(\mu')) \cong \text{Hom}(\text{ind}_{n-1}^\lambda L_{n-1}(\mu'), \text{ind}_{n-1}^\lambda L_{n-1}(\mu')) \neq 0$$

and hence  $L_{n-1}(\mu')$  must appear in the socle of  $\text{res}_n^{\lambda'} \text{ind}_{n-1}^\lambda L_{n-1}(\mu')$ . This forces

$$\text{res}_n^{\lambda'} \text{ind}_{n-1}^\lambda L_{n-1}(\mu') \cong L_{n-1}(\mu')$$

and as we already have that  $\text{res}_n^{\lambda'}$  takes simples to simples we deduce that

$$\text{ind}_{n-1}^\lambda L_{n-1}(\mu') \cong L_n(\mu)$$

which completes our proof of (8). Now (9) follows immediately as  $\text{res}_n^{\lambda'} \Delta_n(\mu) \cong \Delta_{n-1}(\mu')$ , while (10) follows from

$$\begin{aligned} \text{Hom}(\Delta_n(\mu), \Delta_n(\tau)) &\cong \text{Hom}(\text{ind}_{n-1}^\lambda \Delta_{n-1}(\mu'), \Delta_n(\tau)) \\ &\cong \text{Hom}(\Delta_{n-1}(\mu'), \text{res}_n^{\lambda'} \Delta_n(\tau)) \cong \text{Hom}(\Delta_{n-1}(\mu'), \Delta_{n-1}(\tau')). \end{aligned}$$

□

Let  $P_n(\lambda)$  denote the projective cover of  $L_n(\lambda)$ . As our algebras are quasihereditary we have that  $P_n(\lambda)$  has a filtration by standard modules with well-defined filtration multiplicities; we denote the multiplicity of  $\Delta_n(\mu)$  in such a filtration by  $(P_n(\lambda) : \Delta_n(\mu))$ .

**Proposition 4.2.** *Suppose that  $\lambda \in \Lambda_n$  and  $\lambda' \in \Lambda_{n-1}$  are translation equivalent. Then for all  $\mu \in \mathcal{B}_n(\lambda)$  with  $\mu' \in \mathcal{B}_{n-1}(\lambda')$  we have*

$$\text{ind}_{n-1}^\lambda P_{n-1}(\mu') \cong P_n(\mu). \quad (11)$$

If  $\mu \in \mathcal{B}_{n-2}(\lambda)$  we have

$$\text{res}_n^{\lambda'} P_n(\mu) \cong P_{n-1}(\mu'). \quad (12)$$

*Proof.* We begin with (11). The functor  $\text{ind}_{n-1}^\lambda$  takes projectives to projectives, and hence so does  $\text{ind}_{n-1}^\lambda$ . We must show that inducing an indecomposable projective gives an indecomposable projective with the right weight.

Suppose we have an exact sequence

$$\text{ind}_{n-1}^\lambda P_{n-1}(\mu') \rightarrow L_n(\tau) \rightarrow 0$$

for some  $\tau \in \mathcal{B}_n(\lambda)$ . Then we have

$$\begin{aligned} 0 &\neq \text{Hom}_n(\text{ind}_{n-1}^\lambda P_{n-1}(\mu'), L_n(\tau)) \\ &\cong \text{Hom}_{n-1}(P_{n-1}(\mu'), \text{res}_n^{\lambda'} L_n(\tau)) \cong \text{Hom}_{n-1}(P_{n-1}(\mu'), L_n(\tau')) \end{aligned}$$

by Proposition 4.1. Therefore we must have  $\mu' = \tau'$ , and hence  $\mu = \tau$  and

$$\text{Hom}_n(\text{ind}_{n-1}^\lambda P_{n-1}(\mu'), L_n(\mu)) \cong k.$$

This implies that  $\text{ind}_{n-1}^\lambda P_{n-1}(\mu')$  has simple head  $L_n(\mu)$  and hence is isomorphic to  $P_n(\mu)$ .

Next we consider (12). As  $A_n e_n$  is a direct summand of the left  $A_n$ -module  $A_n$ , it is a projective  $A_n$ -module. Moreover, as  $e_n A_n e_n \cong A_{n-2}$  we have that  $A_n e_n$  contains precisely those indecomposable projective  $A_n$ -modules labelled by weights in  $\Lambda_{n-2}$ . In a tower of recollement we have that

$$\text{res}_{n-1} A_n e_n \cong A_{n-1}$$

as a left  $A_{n-1}$ -module. This implies that for  $\mu \in \Lambda_{n-2}$ , the module  $\text{res}_n P_n(\mu)$  (and hence  $\text{res}_n^{\lambda'} P_n(\mu)$ ) is projective.

As  $\text{res}_n^{\lambda'}$  is an exact functor and  $P_n(\mu)$  has simple head  $L_n(\mu)$  we know from Proposition 4.1 that

$$\text{res}_n^{\lambda'} P_n(\mu) = P_n(\mu') \oplus Q \tag{13}$$

for some projective  $A_{n-1}$ -module  $Q$ . However, by Brauer-Humphreys reciprocity and Proposition 4.1 we have

$$(P_n(\mu) : \Delta_n(\tau)) = [\Delta_n(\tau) : L_n(\mu)] = [\Delta_{n-1}(\tau') : L_{n-1}(\mu')] = (P_{n-1}(\mu') : \Delta_{n-1}(\tau')).$$

As  $\text{res}_n^{\lambda'}$  is exact and takes  $\Delta_n(\tau)$  to  $\Delta_{n-1}(\tau')$  this implies that  $Q = 0$ .  $\square$

We would like to argue that two blocks labelled by translation equivalent weights are Morita equivalent. However, the fact that not every projective module restricts to a projective in (12) causes certain complications.

**Lemma 4.3.** *If  $\lambda \in \Lambda_n$  then*

$$G_n(P_n(\lambda)) \cong P_{n+2}(\lambda).$$

*Proof.* By [ASS06, Chapter I, Theorem 6.8]  $G_n(P_n(\lambda))$  is an indecomposable projective. We have an exact sequence

$$P_n(\lambda) \rightarrow \Delta_n(\lambda) \rightarrow 0$$

And hence as  $G_n$  is right exact and takes standards to standards we obtain

$$G_n(P_n(\lambda)) \rightarrow \Delta_{n+2}(\lambda) \rightarrow 0.$$

This implies that  $G_n(P_n(\lambda)) \cong P_{n+2}(\lambda)$ .  $\square$

**Lemma 4.4.** *If  $\mu, \tau \in \Lambda_n$  then*

$$\text{Hom}_n(P_n(\mu), P_n(\tau)) \cong \text{Hom}_{n+2}(P_{n+2}(\mu), P_{n+2}(\tau))$$

*and this extends to an algebra isomorphism*

$$\text{End}_n\left(\bigoplus_{\mu \in \Gamma} P_n(\mu)\right) \cong \text{End}_{n+2}\left(\bigoplus_{\mu \in \Gamma} P_{n+2}(\mu)\right)$$

*where  $\Gamma \subseteq \Lambda_n$ .*

*Proof.* See [ASS06, Chapter I, Theorem 6.8]. □

The algebra  $A_n$  decomposes as a direct sum of indecomposable projective modules:

$$A_n = \bigoplus_{\lambda \in \Lambda_n} P_n(\lambda)^{d_{n,\lambda}}$$

for some integers  $d_{n,\lambda}$ . There is a corresponding decomposition of  $1 \in A_n$  as a sum of (not necessarily primitive) orthogonal idempotents  $1 = \sum_{\lambda \in \Lambda_n} e_{n,\lambda}$  where  $A_n e_{n,\lambda} = P_n(\lambda)^{d_{n,\lambda}}$ . As  $\Lambda_n$  decomposes as a union of blocks the algebra  $A_n$  decomposes as a direct sum of (block) subalgebras

$$A_n = \bigoplus_{\lambda} A_n(\lambda)$$

where the sum runs over a set of block representatives and

$$A_n(\lambda) = \bigoplus_{\mu \in \mathcal{B}_n(\lambda)} P_n(\mu)^{d_{n,\mu}}.$$

Now let  $\Gamma \subset \mathcal{B}_n(\lambda)$  and consider the idempotent  $e_{n,\Gamma} = \sum_{\gamma \in \Gamma} e_{n,\gamma}$ . We define the algebra  $A_{n,\Gamma}(\lambda)$  by

$$A_{n,\Gamma}(\lambda) = e_{n,\Gamma} A_n(\lambda) e_{n,\Gamma}.$$

By Lemma 4.4 we have that  $A_{n,\Gamma}(\lambda)$  and  $A_{m,\Gamma}(\lambda)$  are Morita equivalent for all  $m$  such that  $\Gamma \subset \mathcal{B}_m(\lambda)$ .

**Theorem 4.5.** *Suppose that  $\lambda$  and  $\lambda'$  are translation equivalent with  $\lambda \in \Lambda_n$ , and set*

$$\Gamma = \theta(\mathcal{B}_n(\lambda)) \subset \mathcal{B}_{n+1}(\lambda').$$

*Then  $A_n(\lambda)$  and  $A_{n+1,\Gamma}(\lambda')$  are Morita equivalent. In particular, if there exists an  $n$  such that  $|\mathcal{B}_n(\lambda)| = |\mathcal{B}_{n+1}(\lambda')|$  then  $A_n(\lambda)$  and  $A_{n+1}(\lambda')$  are Morita equivalent.*

*Proof.* We will show that the basic algebras corresponding to  $A_n(\lambda)$  and  $A_{n+1,\Gamma}(\lambda')$  are isomorphic; i.e. that

$$\text{End}_n\left(\bigoplus_{\mu \in \mathcal{B}_n(\lambda)} P_n(\mu)\right) \cong \text{End}_{n+1}\left(\bigoplus_{\nu' \in \Gamma} P_{n+1}(\nu')\right).$$

By Lemma 4.4 it is enough to show that

$$\text{End}_{n+2}\left(\bigoplus_{\mu \in \mathcal{B}_n(\lambda)} P_{n+2}(\mu)\right) \cong \text{End}_{n+1}\left(\bigoplus_{\nu' \in \Gamma} P_{n+1}(\nu')\right).$$

Suppose that  $\mu, \tau \in \mathcal{B}_n(\lambda)$ . Then by Lemma 4.4 and Proposition 4.2 we have

$$\begin{aligned} \mathrm{Hom}_{n+2}(P_{n+2}(\mu), P_{n+2}(\tau)) &\cong \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^\lambda P_{n+1}(\mu'), P_{n+2}(\tau)) \\ &\cong \mathrm{Hom}_{n+1}(P_{n+1}(\mu'), \mathrm{res}_{n+2}^{\lambda'} P_{n+2}(\tau)) \\ &\cong \mathrm{Hom}_{n+1}(P_{n+1}(\mu'), P_{n+1}(\tau')) \end{aligned}$$

Next we will show that these isomorphisms are also compatible with the multiplicative structure in each of our algebras.

Let  $P, Q$ , and  $R$  be indecomposable projectives for  $A_{n+2}$  labelled by elements from  $\mathcal{B}_n(\lambda)$ . Then there exist indecomposable projectives  $P', Q'$  and  $R'$  labelled by elements in  $\mathcal{B}_{n+1}(\lambda')$  such that

$$P = \mathrm{ind}_{n+1}^\lambda P' \quad Q = \mathrm{ind}_{n+1}^\lambda Q' \quad Q' = \mathrm{res}_{n+2}^{\lambda'} Q \quad R' = \mathrm{res}_{n+2}^{\lambda'} R.$$

An isomorphism  $\alpha$  giving a Frobenius reciprocity of the form

$$\mathrm{Hom}_{n+1}(M, \mathrm{res}_{n+2} N) \cong \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1} M, N)$$

is given by the map taking  $\phi$  to  $\alpha(\phi)$  where

$$\alpha(\phi)(a \otimes m) = a\phi(m)$$

for all  $a \in A_{n+2}$  and  $m \in M$ , and extending by linearity. (Recall that  $\mathrm{ind}_{n+1}$  is just the function  $A_{n+2} \otimes_{A_{n+1}} \cdot$ .) Given

$$\begin{array}{ccccc} \mathrm{Hom}_{n+2}(P, Q) & \times & \mathrm{Hom}_{n+2}(Q, R) & \longrightarrow & \mathrm{Hom}_{n+2}(P, R) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^\lambda P', \mathrm{ind}_{n+1}^\lambda Q') \times \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^\lambda Q', R) & \longrightarrow & \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^\lambda P', R) & & \\ \alpha \uparrow & & \alpha \uparrow & & \alpha \uparrow \\ \mathrm{Hom}_{n+1}(P', \mathrm{res}_{n+2}^{\lambda'} \mathrm{ind}_{n+1}^\lambda Q') \times \mathrm{Hom}_{n+1}(Q' \mathrm{res}_{n+2}^{\lambda'} R) & \longrightarrow & \mathrm{Hom}_{n+1}(P' \mathrm{res}_{n+2}^{\lambda'} R) & & \\ \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\ \mathrm{Hom}_{n+1}(P', Q') & \times & \mathrm{Hom}_{n+1}(Q', R') & \longrightarrow & \mathrm{Hom}_{n+1}(P', R') \\ \phi & & \psi & & \psi \circ \phi \end{array} \quad (14)$$

we need to check that  $\alpha(\psi \circ \phi) = \alpha(\psi) \circ \alpha(\phi)$ . We have

$$\alpha(\phi)\left(\sum_i a_i \otimes p_i\right) = \sum_i a_i \phi(p_i)$$

where  $a_i \in A_{n+2}$  and  $p_i \in P'$ . As  $\phi(p_i) \in Q' \cong \mathrm{res}_{n+2}^{\lambda'}(\mathrm{ind}_{n+1}^\lambda Q')$  we have

$$\phi(p_i) = \sum_j a'_j \otimes q_j$$

where  $a'_j \in A_{n+2}$  and  $q_j \in Q'$ . Now

$$\begin{aligned} (\alpha(\psi) \circ \alpha(\phi))\left(\sum_i a_i \otimes p_i\right) &= \alpha(\psi)\left(\sum_i a_i \left(\sum_j a'_j \otimes q_j\right)\right) \\ &= \alpha(\psi)\left(\sum_{i,j} a_i a'_j \otimes q_j\right) = \sum_{i,j} a_i a'_j \psi(q_j) \end{aligned}$$

where the second equality follows from the action of  $A_{n+2}$  on  $\text{ind}_{n-1} Q'$ . On the other hand

$$\begin{aligned} \alpha(\psi \circ \phi) \left( \sum_i a_i \otimes p_i \right) &= \sum_i a_i (\psi \circ \phi)(p_i) \\ &= \sum_i a_i \psi \left( \sum_j a'_j \otimes q_j \right) \\ &= \sum_i a_i \sum_j a'_j \psi(q_j) = \sum_{i,j} a_i a'_j \psi(q_j) \end{aligned}$$

and hence  $\alpha(\psi \circ \phi) = \alpha(\psi) \circ \alpha(\phi)$  as required.  $\square$

Next we will consider how we can relate the cohomology of  $A_{n+1,\Gamma}(\lambda')$  to that of  $A_{n+1}(\lambda')$  and hence compare the cohomology of  $A_n(\lambda)$  with that of  $A_{n+1}(\lambda')$ . We say that a subset  $\Gamma \subset \mathcal{B}_n(\lambda)$  is *saturated* if  $\mu \in \Gamma$  and  $\nu \in \mathcal{B}_n(\lambda)$  with  $\nu > \mu$  implies that  $\nu \in \Gamma$ . A subset  $\Gamma \subset \mathcal{B}_n(\lambda)$  is *cosaturated* if  $\mathcal{B}_n(\lambda) \setminus \Gamma$  is saturated.

**Lemma 4.6.** *The set  $\Gamma = \theta(\mathcal{B}_n(\lambda))$  is cosaturated in  $\mathcal{B}_{n+1}(\lambda')$ .*

*Proof.* We need to show that if  $\mu' \in \mathcal{B}_{n+1}(\lambda') \setminus \Gamma$  and  $\nu' \in \mathcal{B}_{n+1}(\lambda')$  with  $\nu' > \mu'$  then  $\nu' \in \mathcal{B}_{n+1}(\lambda') \setminus \Gamma$ . Suppose for a contradiction that  $\nu' \in \Gamma$ . Then  $\nu = \theta^{-1}(\nu') \in \mathcal{B}_n(\lambda)$  and  $\mu = \theta^{-1}(\mu') \notin \mathcal{B}_n(\lambda)$ . As  $\mu \in \mathcal{B}(\lambda)$  we must have  $|\mu| \geq n+2$ , and as  $\mu' \in \text{supp}(\mu)$  we have

$$|\mu'| \geq n+2 \pm 1 \geq n+1.$$

Now  $|\nu| \leq n$  and so

$$|\nu'| \leq n \pm 1 \leq n+1$$

but this contradicts the assumption that  $\mu' < \nu'$ .  $\square$

If  $\Gamma \subset \mathcal{B}_n(\lambda)$  is cosaturated then by [Don98, A.3.11] the algebra  $A_{n,\Gamma}(\lambda)$  is quasihereditary with standard modules given by

$$\{e_{n,\Gamma} \Delta_n(\mu) : \mu \in \Gamma\}.$$

Moreover if  $X$  is any  $A_n$ -module having a  $\Delta$ -filtration with factors  $\Delta_n(\mu)$  for  $\mu \in \Gamma$ , and  $Y$  is any  $A_n$ -module, then for all  $i \geq 0$  we have [Don98, A.3.13]

$$\text{Ext}_n^i(X, Y) = \text{Ext}_{A_n(\lambda)}^i(X, Y) \cong \text{Ext}_{A_{n,\Gamma}(\lambda)}^i(e_{n,\Gamma} X, e_{n,\Gamma} Y).$$

Combining the above remarks with Theorem 4.5 and Lemma 4.6 we obtain

**Corollary 4.7.** *If  $\lambda \in \Lambda_n$  and  $\lambda' \in \Lambda_{n+1}$  are translation equivalent then for all  $i \geq 0$  and for all  $\mu \in \mathcal{B}_n(\lambda)$  we have*

$$\text{Ext}_n^i(\Delta_n(\lambda), \Delta_n(\mu)) \cong \text{Ext}_{n+1}^i(\Delta_{n+1}(\lambda'), \Delta_{n+1}(\mu')).$$

We will say that two weights  $\lambda$  and  $\lambda'$  are in the same *translation class* if they are related by the equivalence relation generated by translation equivalence. Then analogues of (9), (10), Theorem 4.5 and Corollary 4.7 also hold for weights in the same translation class.

In Lie theory one can consider translation between two weights in the same facet (corresponding to the case considered above), or from one facet to another. We will now give an analogue of translation onto a wall for a tower of recollement.

We will say that  $\lambda'$  *separates*  $\lambda^-$  and  $\lambda^+$  if

- (i) The weight  $\lambda'$  is the only element of  $\mathcal{B}(\lambda') \cap \text{supp}(\lambda^-)$ .
- (ii) The weight  $\lambda'$  is the only element of  $\mathcal{B}(\lambda') \cap \text{supp}(\lambda^+)$ .
- (iii) The weights  $\lambda^+$  and  $\lambda^-$  are the only elements of  $\mathcal{B}(\lambda^-) \cap \text{supp}(\lambda')$ .

Whenever we consider a pair of weights  $\lambda^-$  and  $\lambda^+$  separated by  $\lambda'$  we shall always assume that  $\lambda^- < \lambda^+$ .

**Theorem 4.8.** (i) If  $\lambda' \in \Lambda_{n-1}$  separates  $\lambda^-$  and  $\lambda^+$  then

$$\text{res}_n^{\lambda'} L_n(\lambda^+) \cong L_{n-1}(\lambda').$$

(ii) If further we have  $\text{Hom}(\Delta_n(\lambda^+), \Delta_n(\lambda^-)) \neq 0$  then

$$\text{res}_n^{\lambda'} L_n(\lambda^-) = 0$$

and  $\text{ind}_{n-1}^{\lambda^-} \Delta_n(\lambda')$  is a nonsplit extension of  $\Delta_n(\lambda^-)$  by  $\Delta_n(\lambda^+)$  and has simple head  $L_n(\lambda^+)$ .

*Proof.* Arguing as in the proof of Proposition 4.1 we see that  $\text{res}_n^{\lambda'} L_n(\lambda^i)$  is either 0 or has simple head  $L_{n-1}(\lambda')$  for  $i = +, -$ . Also, any other composition factors  $L_{n-1}(\tau')$  of  $\text{res}_n^{\lambda'} L_n(\lambda^i)$  must satisfy  $\tau' > \lambda'$ . Note that by assumption we have a short exact sequence

$$0 \longrightarrow \Delta_n(\lambda^-) \longrightarrow \text{ind}_{n-1}^{\lambda^-} \Delta_{n-1}(\lambda') \longrightarrow \Delta_n(\lambda^+) \longrightarrow 0 \quad (15)$$

and hence

$$\text{Hom}(\Delta_{n-1}(\tau'), \text{res}_n^{\lambda'} L_n(\lambda^i)) \cong \text{Hom}(\text{ind}_{n-1}^{\lambda^-} \Delta_{n-1}(\tau'), L_n(\lambda^i)) \quad (16)$$

is non-zero when  $i = +$  and  $\tau' = \lambda'$ , and is zero when  $\tau' > \lambda'$  by our assumptions. This completes the proof of (i).

Now suppose that  $\text{Hom}(\Delta_n(\lambda^+), \Delta_n(\lambda^-)) \neq 0$ . Then we have that

$$[\Delta_n(\lambda^-) : L_n(\lambda^+)] \neq 0.$$

By exactness and the first part of the Theorem, the unique copy of  $L_{n-1}(\lambda')$  in

$$\text{res}_n^{\lambda'} \Delta_n(\lambda^-) \cong \Delta_{n-1}(\lambda')$$

must come from  $\text{res}_n^{\lambda'} L_n(\lambda^+)$ , and hence  $\text{res}_n^{\lambda'} L_n(\lambda^-)$  cannot have simple head  $L_{n-1}(\lambda')$ . But this implies by the first part of the proof that  $\text{res}_n^{\lambda'} L_n(\lambda^-) = 0$ . Therefore the Hom-space in (16) must be zero when  $\tau' = \lambda'$  and  $i = 1$ , which implies that (15) is a non-split extension whose central module has simple head  $L_n(\lambda^+)$  as required.  $\square$

Suppose that  $\lambda'$  and  $\lambda^+$  are weights with  $\lambda' < \lambda^+$  and  $\lambda' \in \text{supp}(\lambda^+)$  such that for every weight  $\tau' \in \mathcal{B}(\lambda')$  either (i) there is a unique weight  $\tau^+ \in \mathcal{B}(\lambda^+) \cap \text{supp}(\tau')$  and  $\tau'$  is the unique weight in  $\mathcal{B}(\lambda') \cap \text{supp}(\tau^+)$ , or (ii) there exists  $\tau^-, \tau^+ \in \mathcal{B}(\lambda^+)$  such that  $\tau'$  separates  $\tau^-$  and  $\tau^+$ . Then we say that  $\lambda'$  is in the *lower closure* of  $\lambda^+$ . If further

$$\text{Hom}(\Delta_n(\tau^+), \Delta_n(\tau^-)) \neq 0$$

whenever  $\tau' \in \mathcal{B}(\lambda')$  separates  $\tau^-$  and  $\tau^+$  in  $\mathcal{B}(\lambda^+)$  then we shall say that  $\mathcal{B}(\lambda^+)$  has *local homomorphisms* with respect to  $\mathcal{B}(\lambda')$ .

**Proposition 4.9.** Suppose that  $\lambda' \in \Lambda_{n-1}$  is in the lower closure of  $\lambda^+ \in \Lambda_n$ , and that  $\mathcal{B}(\lambda^+)$  has enough local homomorphisms with respect to  $\mathcal{B}(\lambda')$ . Then

$$[\Delta_{n-1}(\lambda') : L_{n-1}(\mu')] = [\Delta_n(\lambda^+) : L_n(\mu^+)].$$

*Proof.* We have by our assumptions that

$$\mathrm{res}_n^{\lambda'} \Delta_n(\lambda^+) \cong \Delta_{n-1}(\lambda').$$

As  $\mathrm{res}_n^{\lambda'}$  is an exact functor, it is enough to determine its effect on simples  $L_n(\mu^+)$  in  $\Delta_n(\lambda^+)$ . If there exists  $\mu'$  separating  $\mu^+$  from  $\mu^-$  then the result follows from Theorem 4.8, while if  $\mu^+$  is the only element in  $\mathcal{B}(\lambda^+) \cap \mathrm{supp}(\mu')$  then it follows as in the proof of Proposition 4.1.  $\square$

Thus, as long as there are enough local homomorphisms, the decomposition numbers for  $\Delta_n(\lambda)$  determine those for all weights in the lower closure of  $\lambda$ .

We can generalise the results of this section up to Corollary 4.7 by replacing  $\mathrm{res}_n^\lambda$  and  $\mathrm{ind}_n^\lambda$  by any pair of functor families  $R_n$  and  $I_n$  with the following properties.

Let  $A$  be a tower of recollement, with  $\lambda, \lambda' \in \Lambda$ , and fix  $i \in \mathbb{N}$ . Suppose that we have functors

$$R_n : A_n\text{-mod} \rightarrow A_{n-i}\text{-mod}$$

for  $n \geq i$  and

$$I_n : A_n\text{-mod} \rightarrow A_{n+i}\text{-mod}$$

for  $n \geq 0$  satisfying

- (i) The functor  $I_n$  is left adjoint to  $R_{n+i}$  for all  $n$ .
- (ii) The functor  $R_n$  is exact and  $I_n$  is right exact for all  $n$  where they are defined.
- (iii) There is a bijection  $\theta : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda')$  taking  $\mu$  to  $\mu'$  such that for all  $n \geq i$ , if  $\mu \in \mathcal{B}_n(\lambda)$  and  $\mu' \in \mathcal{B}_{n-i}(\lambda')$  then

$$R_n \Delta_n(\mu) \cong \Delta_{n-i}(\mu') \quad \text{and} \quad I_{n-i} \Delta_{n-i}(\mu') = \Delta_n(\mu)$$

and  $R_n \Delta_n(\mu) = 0$  otherwise.

- (iv) If  $\Gamma_n = \theta(\mathcal{B}_n(\lambda)) \subset \mathcal{B}_m(\lambda')$  for some  $m$  then  $\Gamma_n$  is cosaturated in  $\mathcal{B}_m(\lambda')$ .
- (v) There exists  $t \in \mathbb{N}$  such that for all  $n$  and for all  $\mu \in \mathcal{B}_{n-t}(\lambda)$  the module  $R_n P_n(\mu)$  is projective.

Then we say that  $\lambda$  and  $\lambda'$  are  $(R, I)$ -translation equivalent. In this case the proofs of Propositions 4.1 and 4.2 go through essentially unchanged, and we get

**Theorem 4.10.** *Suppose that  $\lambda$  and  $\lambda'$  are  $(R, I)$ -translation equivalent and  $n \geq i$ . Then for all  $\mu \in \mathcal{B}_n(\lambda)$  with  $\mu' \in \mathcal{B}_{n-i}(\lambda')$  we have*

$$R_n L_n(\mu) \cong L_{n-i}(\mu'), \quad I_{n-i} L_{n-i}(\mu') \cong L_n(\mu) \quad \text{and} \quad I_{n-i} P_{n-i}(\mu') \cong P_n(\mu)$$

and if  $\mu \in \mathcal{B}_{n-t}(\lambda)$  then

$$R_n P_n(\mu) \cong P_{n-i}(\mu').$$

Moreover, if the adjointness isomorphism

$$\alpha : \mathrm{Hom}_{n-i}(M, R_n(N)) \rightarrow \mathrm{Hom}_n(I_{n-i}(M), N)$$

is multiplicative (i.e. makes the diagram (14) commute) then there is a Morita equivalence between  $A_n(\lambda)$  and  $A_{n+i, \Gamma_{n+i}}(\lambda')$  and for all  $\mu, \tau \in \mathcal{B}_n(\lambda)$  and  $j \geq 0$  we have

$$\mathrm{Ext}_n^j(\Delta_n(\mu), \Delta_n(\tau)) \cong \mathrm{Ext}_{n+i}^j(\Delta_{n+i}(\mu'), \Delta_{n+i}(\tau')).$$

## 5. A GENERALISED RESTRICTION/INDUCTION PAIR

We wish to show (in Section 6) that two weights in the same facet for the Brauer algebra give rise to Morita equivalent blocks (at least when we truncate the blocks to have the same number of simples). However, the usual induction and restriction functors are not sufficient to show this except in the alcove case. To remedy this, in this section we will consider a variation on the usual induction and restriction functors. As  $\delta$  will be fixed throughout, we will denote  $B_n(\delta)$  simply as  $B_n$ .

First consider  $B_2$  with  $\delta \neq 0$ . It is easy to see that this is a semisimple algebra, with a decomposition

$$1 = e + e^- + e^+$$

of the identity into primitive orthogonal idempotents given by the elements in Figure 1.

$$\begin{aligned} e &= \frac{1}{\delta} \left( \begin{array}{c} \cup \\ \cap \end{array} \right) & e^- &= \frac{1}{2} \left( \begin{array}{c} | \\ | \\ | \end{array} - \begin{array}{c} \diagup \diagdown \end{array} \right) \\ e^+ &= \frac{1}{2} \left( \begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} \diagup \diagdown \end{array} \right) - \frac{1}{\delta} \left( \begin{array}{c} \cup \\ \cap \end{array} \right) \end{aligned}$$

FIGURE 1. Idempotents in  $B_2$

There are three standard modules for this algebra, which we will denote by

$$\begin{aligned} \Delta_2(0) &= \langle e \rangle \\ S^- &= \Delta_2(1, 1) = \langle e^- \rangle \\ S^+ &= \Delta_2(2) = \langle e^+ \rangle \end{aligned}$$

For  $n \geq 2$  consider the subalgebra  $B_{n-2} \otimes B_2 \subseteq B_n$  obtained by letting  $B_{n-2}$  act on the leftmost  $n-2$  lines and  $B_2$  act on the rightmost pair of lines. We will view elements of  $B_{n-2}$  and  $B_2$  as elements of  $B_n$  via this embedding. Note that under this embedding the two algebras obviously commute with each other.

In particular, for any  $B_n$ -module  $M$  the vector spaces  $e^\pm M$  are  $B_{n-2}$ -modules. Thus we have a pair of functors  $\text{res}_n^\pm$  from  $B_n\text{-mod}$  to  $B_{n-2}\text{-mod}$  given on objects by the map  $M \mapsto e^\pm M$ . Note that these functors can also be defined as

$$\text{res}_n^\pm M = e^\pm \text{res}_{B_{n-2} \otimes B_2}^{B_n} M.$$

We have

$$\begin{aligned} \text{Hom}_{B_n}(\text{ind}_{B_{n-2} \otimes B_2}^{B_n}(N \boxtimes S^\pm), M) &\cong \text{Hom}_{B_{n-2} \otimes B_2}(N \boxtimes S^\pm, \text{res}_{B_{n-2} \otimes B_2}^{B_n} M) \\ &\cong \text{Hom}_{B_{n-2}}(N, e^\pm \text{res}_{B_{n-2} \otimes B_2}^{B_n} M) \end{aligned}$$

and so the functors  $\text{ind}_{n-2}^\pm$  from  $B_{n-2}\text{-mod}$  to  $B_n\text{-mod}$  given by

$$\text{ind}_{n-2}^\pm N = \text{ind}_{B_{n-2} \otimes B_2}^{B_n}(N \boxtimes S^\pm)$$

are left adjoint to  $\text{res}_n^\pm$ .

**Lemma 5.1.** *Let  $N$  be a  $B_{n-2}$ -module. Then we have*

$$\text{ind}_{n-2}^\pm N \cong B_n e^\pm \otimes_{B_{n-2}} N$$

as  $B_n$ -modules, where the action on the right-hand space is by left multiplication in  $B_n$ .

*Proof.* Define a map

$$\phi : B_n \otimes_{B_{n-2} \otimes B_2} (N \boxtimes S^\pm) \rightarrow B_n e^\pm \otimes_{B_{n-2}} N$$

by

$$b \otimes (n \otimes e^\pm) \mapsto b e^\pm \otimes n.$$

We first show that this is well-defined. Let  $b = b' b_{n-2} b_2$  for some  $b_{n-2} \in B_{n-2}$  and  $b_2 \in B_2$ . Then

$$\begin{aligned} \phi(b \otimes (n \otimes e^\pm) - b' \otimes (b_{n-2} n \otimes b_2 e^\pm)) &= b' b_{n-2} b_2 e^\pm \otimes n - b' b_2 e^\pm \otimes b_{n-2} n \\ &= b' b_2 e^\pm b_{n-2} \otimes n - b' b_2 e^\pm \otimes b_{n-2} n = 0 \end{aligned}$$

as required. The map  $\phi$  is clearly a  $B_n$ -homomorphism. We also have a map

$$\psi : B_n e^\pm \otimes_{B_{n-2}} N \rightarrow B_n \otimes_{B_{n-2} \otimes B_2} (N \boxtimes S^\pm)$$

given by

$$b e^\pm \otimes n \mapsto b e^\pm \otimes (n \otimes e^\pm).$$

It is easy to check that  $\psi$  is well-defined and that  $\psi\phi = \text{id}$  and  $\phi\psi = \text{id}$ . □

Let  $e_{n,4}$  be the idempotent in  $B_n$  shown in Figure 2.

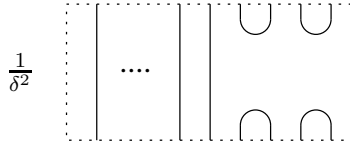


FIGURE 2. An idempotent in  $B_n$

**Lemma 5.2.** *As left  $B_n$ - and right  $B_{n-2}$ -modules we have*

$$e^\pm B_{n+2} e_{n+2,4} \cong B_n e^\pm.$$

*Proof.* Consider the map from  $e^\pm B_{n+2} e_{n+2,4}$  to  $B_n e^\pm$  given on diagrams as shown in Figure 3. The grey shaded regions show the actions of  $B_n$  from above, of  $B_{n-2}$  from below, and the dark shaded region the action of the element  $e^\pm$ . All lines in the diagrams except those indicated remain unchanged; the two southern arcs in the left-hand diagram are removed, and the ends of the pair of lines acted on by  $e^\pm$  are translated clockwise around the boundary from the northern to the southern side. This gives an isomorphism of vector spaces, and clearly preserves the actions of  $B_n$  and  $B_{n-2}$ . □

**Corollary 5.3.** *The module  $\text{res}_n^\pm(P_n(\lambda))$  is projective for all  $\lambda \in \Lambda_{n-4}$ .*

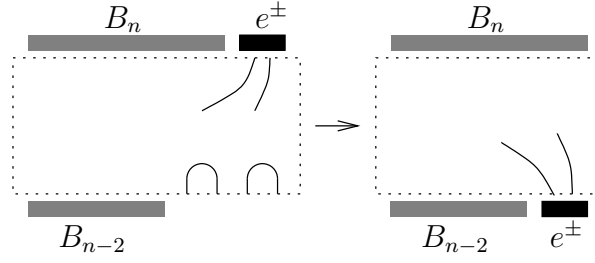


FIGURE 3. Realising the isomorphism between  $e^{\pm} B_{n+2} e_{n+2,4}$  and  $B_n e^{\pm}$

*Proof.* First note that  $B_n e_{n,4}$  is a projective  $B_n$ -module. Moreover, as  $e_{n,4} B_n e_{n,4} \cong B_{n-4}$  we have that  $B_n e_{n,4}$  contains precisely the indecomposable projectives labelled by elements of  $\Lambda_{n-4}$ . By Lemma 5.2 we have that  $e^{\pm} B_n e_{n,4} \cong B_{n-2} e^{\pm}$  as left  $B_{n-2}$ -modules, and hence  $\text{res}_n^{\pm}(P_n(\lambda))$  is projective for all  $\lambda \in \Lambda_{n-4}$ .  $\square$

**Corollary 5.4.** *We have an isomorphism of functors*

$$\text{ind}_n^{\pm} \cong \text{res}_{n+4}^{\pm} G_{n+2} G_n.$$

*Proof.* By the definition of  $G_n$  and  $G_{n+2}$  we have

$$\begin{aligned} \text{res}_{n+4}^{\pm} G_{n+2} G_n(N) &= \text{res}_{n+4}^{\pm} (B_{n+4} e_{n+4,2} \otimes_{B_n} N) \\ &= e^{\pm} B_{n+4} e_{n+4,4} \otimes_{B_n} N \\ &\cong B_{n+2} e^{\pm} \otimes_{B_n} N \end{aligned}$$

where the final isomorphism follows from Lemma 5.2. But by Lemma 5.1 this final module is isomorphic to  $\text{ind}_n^{\pm} N$ .  $\square$

Corollary 5.4 is an analogue of the relation between induction, restriction and globalisation in [CDM05, Lemma 2.6(ii)], corresponding to axiom (A4) for a tower of recollement.

Given two partitions  $\lambda$  and  $\mu$ , we write  $\lambda \triangleright \triangleright^+ \mu$ , or  $\mu \triangleleft \triangleleft^+ \lambda$ , if  $\mu$  can be obtained from  $\lambda$  by removing two boxes and  $\lambda/\mu$  is not the partition  $(1, 1)$ . Similarly we write  $\lambda \triangleright \triangleright^- \mu$ , or  $\mu \triangleleft \triangleleft^- \lambda$  if  $\mu$  can be obtained from  $\lambda$  by removing two boxes and  $\lambda/\mu \neq (2)$ . We will write  $\lambda \triangleleft \triangleright \mu$  if  $\mu$  is obtained from  $\lambda$  by removing a box and then adding a box.

The next theorem describes the structure of  $\text{res}_n^{\pm} \Delta_n(\lambda)$ , and so is an analogue of the usual induction and restriction rules in [DWH99, Theorem 4.1 and Corollary 6.4] (and use the same strategy for the proof).

**Theorem 5.5.** *Suppose that  $\lambda$  is a partition of  $m = n - 2t$  for some  $t \geq 0$ .*

(i) *There is a filtration of  $B_{n-2}$ -modules*

$$W_0 \subseteq W_1 \subseteq W_2 = \text{res}_n^{\pm} \Delta_n(\lambda)$$

with

$$W_0 \cong \bigoplus_{\mu \triangleleft \triangleleft^{\pm} \lambda} \Delta_{n-2}(\lambda) \quad W_1/W_0 \cong \bigoplus_{\mu \triangleleft \triangleright \lambda} \Delta_{n-2}(\lambda) \quad W_2/W_1 \cong \bigoplus_{\mu \triangleright \triangleright^{\pm} \lambda} \Delta_{n-2}(\lambda)$$

where any  $\Delta_{n-2}(\mu)$  which does not make sense is taken as 0.

(ii) There is a filtration of  $B_{n+2}$ -modules

$$U_0 \subseteq U_1 \subseteq U_2 = \text{ind}_n^\pm \Delta_n(\lambda)$$

with

$$U_0 \cong \bigoplus_{\mu \triangleleft \triangleleft^\pm \lambda} \Delta_{n+2}(\lambda) \quad U_1/U_0 \cong \bigoplus_{\mu \triangleleft \triangleright \lambda} \Delta_{n+2}(\lambda) \quad U_2/U_1 \cong \bigoplus_{\mu \triangleright \triangleright^\pm \lambda} \Delta_{n+2}(\lambda).$$

*Proof.* Part (ii) follows from part (i) by Corollary 5.4. For the rest of the proof we will work with the concrete realisation of standard modules given in Section 2. By definition we have

$$\text{res}_n^\pm \Delta_n(\lambda) = e^\pm I_n^t \otimes_{\Sigma_m} S^\lambda$$

and we will represent an element  $e^\pm X_{w,1,id} \otimes x$  in this space diagrammatically as shown in Figure 4.

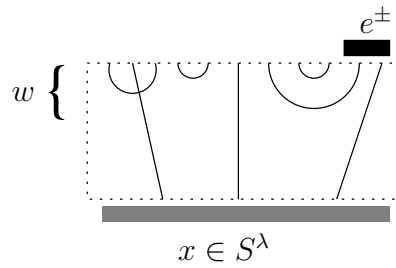


FIGURE 4. Representing the element  $e^\pm X_{w,1,id} \otimes x$  in  $e^\pm I_n^t \otimes_{\Sigma_m} S^\lambda$

We are now in a position to define the various spaces  $W_0$ ,  $W_1$ , and  $W_2$ . Choose a fixed basis  $V(\lambda)$  for  $S^\lambda$  and set

$$V_{n,t}^0 = \{w \in V_{n,t} : n-1 \text{ and } n \text{ are free in } w\}$$

$$V_{n,t}^1 = \{w \in V_{n,t} : n-1 \text{ is on an arc and } n \text{ is free in } w\}$$

$$V_{n,t}^2 = \{w \in V_{n,t} : n-1 \text{ is linked to } j \text{ and } n \text{ is linked to } i \text{ in } w \text{ with } i < j \leq n-2\}.$$

Then for  $0 \leq i \leq 2$  we set

$$W_i = \text{span}\{e^\pm X_{w,1,id} \otimes x : w \in V_{n,t}^j \text{ with } j \leq i \text{ and } x \in V(\lambda)\}$$

Note that if  $w \in V_{n,t}^1 \cup V_{n,t}^2$  and  $w'$  is obtained from  $w$  by swapping nodes  $n-1$  and  $n$ , then

$$e^\pm X_{w,1,id} = \pm e^\pm X_{w',1,id}.$$

Moreover, if there is an arc linking nodes  $n-1$  and  $n$  in  $w$  then

$$e^\pm X_{w,1,id} = 0.$$

Thus we have that

$$W_2 = e^\pm \Delta_n(\lambda)$$

and  $W_0$  and  $W_1$  are submodules of  $e^\pm \Delta_n(\lambda)$ .

We first show that

$$W_0 \cong I_{n-2}^t \otimes_{\Sigma_{m-2}} \sigma^\pm S^\lambda \tag{17}$$

where  $\sigma^\pm$  represents the symmetriser/antisymmetriser on the last two lines in  $\Sigma_m$  and  $\Sigma_{m-2} \subset \Sigma_m$  acts on the first  $m-2$  lines. Note that

$$\begin{aligned} \sigma^\pm S^\lambda &= \sigma^\pm \operatorname{res}_{\Sigma_{m-2} \times \Sigma_2}^{\Sigma_m} S^\lambda \\ &= \sigma^\pm \left( \bigoplus_{\mu \vdash m-2, \nu \vdash 2} c_{\mu, \nu}^\lambda (S^\mu \boxtimes S^\nu) \right) = \bigoplus_{\mu \vdash m-2} c_{\mu, *}^\lambda S^\mu \end{aligned}$$

where  $*$  equals (2) for  $\sigma^+$  and (1, 1) for  $\sigma^-$ . As

$$c_{\mu, (2)}^\lambda = \begin{cases} 1 & \text{if } \mu \triangleleft \triangleleft^+ \lambda \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c_{\mu, (1,1)}^\lambda = \begin{cases} 1 & \text{if } \mu \triangleleft \triangleleft^- \lambda \\ 0 & \text{otherwise} \end{cases}$$

it will follow from (1) that

$$W_0 \cong \bigoplus_{\mu \triangleleft \triangleleft^\pm \lambda} \Delta_{n-2}(\lambda)$$

as required.

Note that for  $w \in V_{n,t}^0$  the lines from  $n-1$  and  $n$  are propagating in  $X_{w,1,id}$ , and so we have

$$e^\pm X_{w,1,id} \otimes S^\lambda = X_{w,1,id} e^\pm \otimes S^\lambda = X_{w,1,id} \otimes \sigma^\pm S^\lambda.$$

For  $w \in V_{n,t}^0$  define  $\bar{w} \in V_{n-2,t}$  by removing nodes  $n-1$  and  $n$ , and a map

$$\phi_0 : W_0 \rightarrow I_{n-2}^t \otimes_{\Sigma_{m-2}} \sigma^\pm S^\lambda$$

by

$$e^\pm X_{w,1,id} \otimes x = X_{w,1,id} \otimes \sigma^\pm x \mapsto X_{\bar{w},1,id} \otimes \sigma^\pm x.$$

It is clear that  $\phi_0$  is an isomorphism of vector spaces, and commutes with the action of  $B_{n-2}$ . This proves (17).

Next we will show that

$$W_1/W_0 \cong I_{n-2}^{t-1} \otimes_{\Sigma_m} \operatorname{ind}_{\Sigma_{m-1}}^{\Sigma_m} \operatorname{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda. \quad (18)$$

Note that

$$\operatorname{ind}_{\Sigma_{m-1}}^{\Sigma_m} \operatorname{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda = \operatorname{ind}_{\Sigma_{m-1}}^{\Sigma_m} \left( \bigoplus_{\nu \triangleleft \lambda} S^\nu \right) = \bigoplus_{\nu \triangleleft \lambda} (\operatorname{ind}_{\Sigma_{m-1}}^{\Sigma_m} S^\nu) = \bigoplus_{\mu \triangleleft \triangleright \lambda} S^\mu$$

and so it will follow from (1) that

$$W_1/W_2 \cong \bigoplus_{\mu \triangleleft \triangleright \lambda} \Delta_{n-2}(\lambda)$$

as required.

We will need an explicit description of  $\operatorname{ind}_{\Sigma_{m-1}}^{\Sigma_m} \operatorname{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$ . The quotient  $\Sigma_m/\Sigma_{m-1}$  has coset representatives

$$\{\tau_i = (i, m) : 1 \leq i \leq m\}$$

where  $(m, m) = 1$ . Therefore  $\operatorname{ind}_{\Sigma_{m-1}}^{\Sigma_m} \operatorname{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$  has a basis

$$\{(i, x) : 1 \leq i \leq m, x \in V(\lambda)\}$$

and the action of  $\theta \in \Sigma_m$  is given by

$$\theta(i, x) = (j, \theta'x)$$

where  $\theta\tau_i = \tau_j\theta'$  for a unique  $1 \leq j \leq m$  and  $\theta' \in \Sigma_{m-1}$ .

For  $1 \leq i \leq m$  set

$$\sigma_i = (i, m, m-1, m-2, \dots, i+1)$$

and for  $w \in V_{n,t}^1$  define  $\bar{w} \in V_{n-2,t-1}$  by removing the nodes  $n-1$  and  $n$  and removing the arc from  $n-1$  (which will thus introduce a new free node elsewhere in  $\bar{w}$ ). Now we can define a map

$$\phi_1 : W_1/W_0 \rightarrow I_{n-2}^{t-1} \otimes_{\Sigma_m} \text{ind}_{\Sigma_{m-1}}^{\Sigma_m} \text{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$$

by

$$e^\pm X_{w,1,id} \otimes x \mapsto X_{\bar{w},1,id} \sigma_i \otimes (m, x)$$

if node  $n-1$  is linked to node  $i$  in  $w$ . This is illustrated graphically in Figure 5.

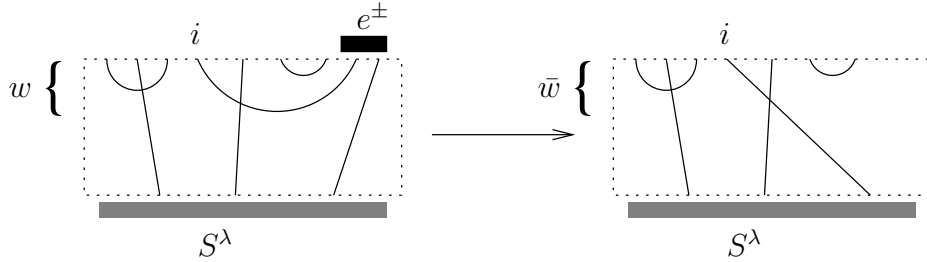


FIGURE 5. An example of the effect of the map  $\phi_1$

Note that for every  $v \in V_{n-2,t-1}$  there are exactly  $m$  elements  $w \in V_{n,t}^1$  satisfying  $\bar{w} = v$ , as  $n-1$  can be joined to any of the  $m$  free vertices in  $v$ . Note also that

$$\sigma_i = (i, m)(m-1, m-2, \dots, i+1, i) = (i, m)\sigma'_i$$

where  $\sigma'_i \in \Sigma_{m-1}$ , and so  $\sigma_i(m, x) = (i, \sigma'_i x)$ .

Given  $v \in V_{n-2,t-1}$ ,  $1 \leq i \leq m$ , and  $x \in V(\lambda)$  pick  $w \in V_{n,t}^1$  with  $\bar{w} = v$  and  $n-1$  joined to the  $i$ th free node. Then

$$\begin{aligned} \phi_1(e^\pm X_{w,1,id} \otimes (\sigma'_i)^{-1}x) &= X_{v,1,id} \sigma_i \otimes (m, (\sigma'_i)^{-1}x) \\ &= X_{v,1,id} \otimes (i, \sigma'_i(\sigma'_i)^{-1}x) = X_{v,1,id} \otimes (i, x) \end{aligned}$$

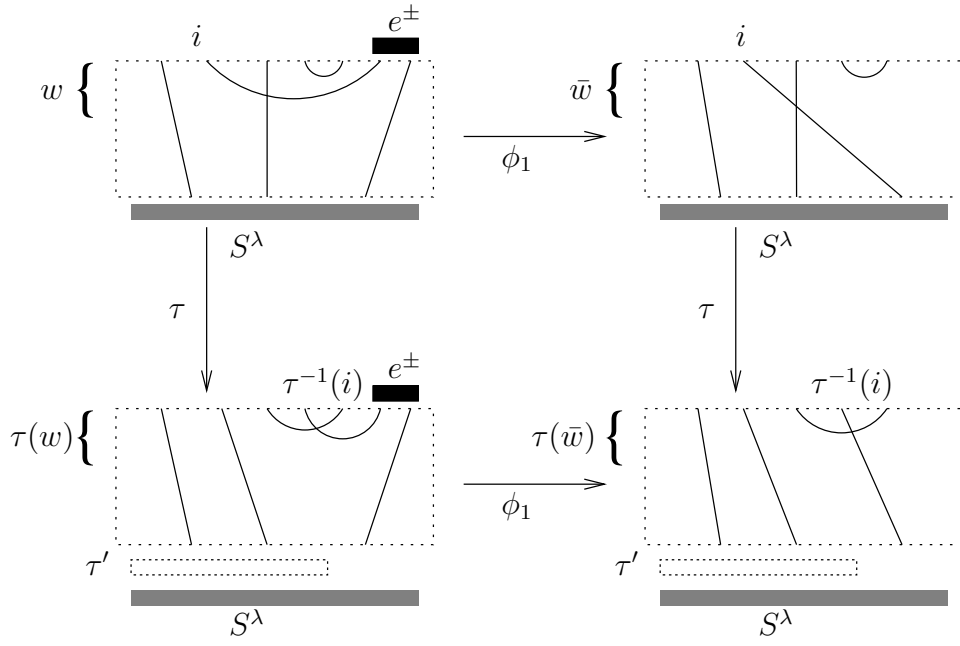
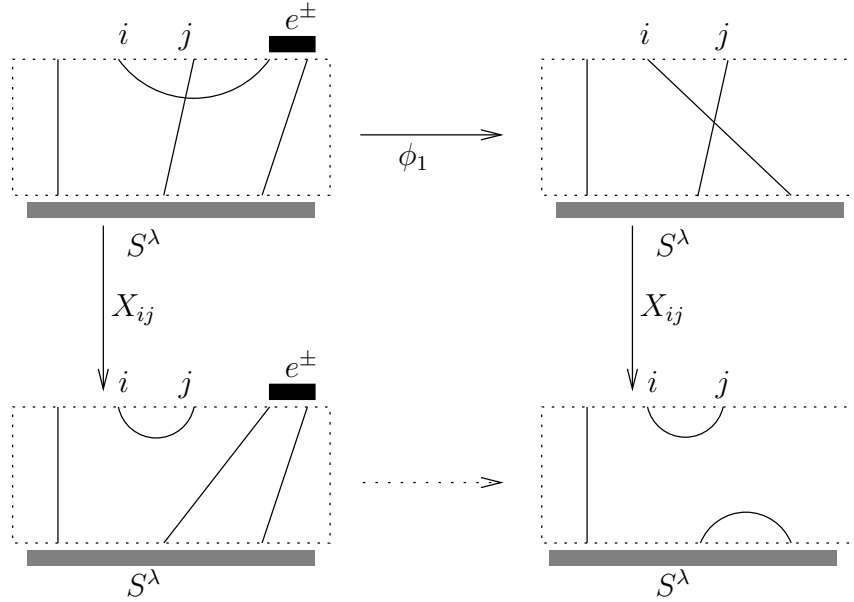
and so  $\phi_1$  is surjective. Moreover

$$\dim W_1/W_2 = m|V_{n-2,t-1}| \dim S^\lambda = \dim I_{n-2}^{t-1} \otimes \text{ind}_{\Sigma_{m-1}}^{\Sigma_m} \text{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$$

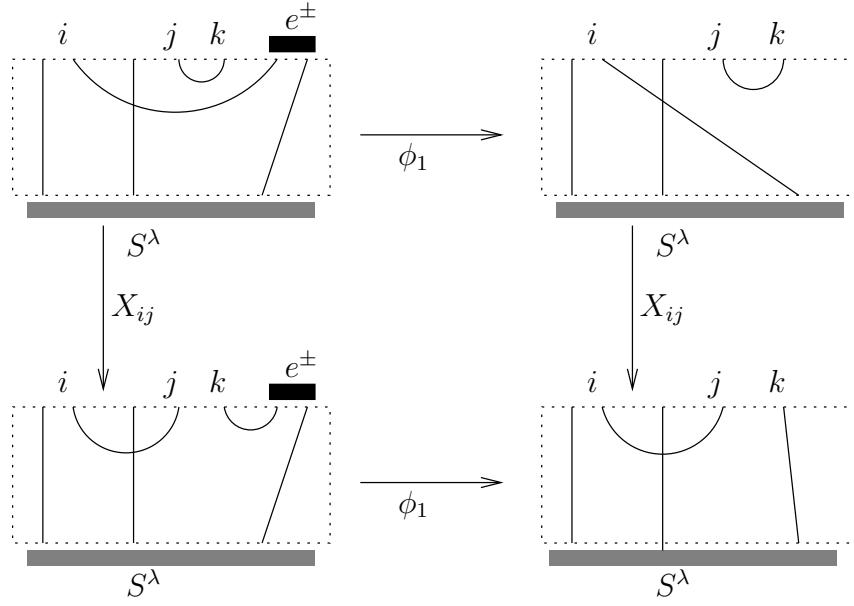
and so  $\phi_1$  is an isomorphism of vector spaces. It remains to show that  $\phi_1$  commutes with the action of  $B_{n-2}$ .

First consider the action of  $\tau \in \Sigma_{n-2}$ . The actions of  $\phi_1$  and  $\tau$  can be seen to commute by the schematic diagram in Figure 6, noting that  $\tau(\bar{w}) = \tau(w)$ .

Next consider the action of  $X_{jk} \in B_{n-2}$ . If  $j, k \neq i$  then it is clear that  $X_{jk}$  commutes with  $\phi_1$ . Now consider the action of  $X_{ij}$ . There are two cases: (i)  $j$  is a free node in  $w$ , and (ii)  $j$  is linked to some node  $k$  in  $w$ . Case (i) is illustrated schematically in Figure 7. The lower left diagram in Figure 7 represents 0 as it lies in  $W_0$ . The lower right diagram represents 0 as there is a decrease in the number of propagating lines. Therefore the dotted arrow is an equality and the diagram commutes.

FIGURE 6. A diagrammatic illustration that  $\phi_1 \tau = \tau \phi_1$ FIGURE 7. The action of  $X_{ij}$  and  $\phi_1$ : case (i)

Case (ii) is illustrated schematically in Figure 8. Again we see that  $X_{ij}$  commutes with the action of  $\phi_1$ , and so we have shown that  $\phi_1$  is a  $B_{n-2}$ -isomorphism. This completes the proof of (18).

FIGURE 8. The action of  $X_{ij}$  and  $\phi_1$ : case (ii)

Finally, we shall show that

$$W_2/W_1 \cong I_{n-2}^{t-2} \otimes_{\Sigma_{m-2}} \text{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}} (S^\lambda \boxtimes S^\pm). \quad (19)$$

As

$$\text{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}} (S^\lambda \boxtimes S^\pm) \cong \bigoplus_{\mu \triangleright \triangleright^\pm \lambda} S^\mu$$

it will follow from (1) that

$$W_2/W_1 \cong \bigoplus_{\mu \triangleright \triangleright^\pm \lambda} \Delta_{n-2}(\lambda)$$

which will complete the proof.

We will need an explicit description of  $\text{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}} (S^\lambda \boxtimes S^\pm)$ . The quotient  $\Sigma_{m+2}/(\Sigma_m \times \Sigma_2)$  has coset representatives

$$\{\tau_{ij} = (i, m+1)(j, m+2) : 1 \leq i < j \leq m+2\}$$

where  $(m+1, m+1) = (m+2, m+2) = 1$ . Therefore  $\text{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}} (S^\lambda \boxtimes S^\pm)$  has a basis

$$\{(i, j; x \otimes \sigma^\pm) : 1 \leq i < j \leq m+2, x \in V(\lambda)\}$$

and the action of  $\theta \in \Sigma_{m+2}$  is given by

$$\theta(i, j; x \otimes \sigma^\pm) = (k, l; \theta'(x \otimes \sigma^\pm))$$

where  $\theta\tau_{ij} = \tau_{kl}\theta'$  for a unique  $1 \leq k < l \leq m+2$  and  $\theta' \in \Sigma_m \times \Sigma_2$ .

For  $1 \leq r < s \leq m+2$  set

$$\sigma_{r,s} = (r, s, s-1, s-2, \dots, r+1)$$

and for  $w \in V_{n,t}^2$  define  $\bar{w} \in V_{n-2,t-2}$  by removing the nodes  $n-1$  and  $n$  and removing the arcs from  $n-1$  and  $n$  (which will thus introduce two new free nodes elsewhere in  $\bar{w}$ ). Now we can define a map

$$\phi_2 : W_2/W_1 \rightarrow I_{n-2}^{t-2} \otimes_{\Sigma_{m+2}} \text{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}} (S^\lambda \boxtimes S^\pm)$$

by

$$e^\pm X_{w,1,id} \otimes x \longmapsto X_{\bar{w},1,id} \sigma_{j,m+2} \sigma_{i,m+1} \otimes (m+1, m+2; , x \otimes \sigma^\pm)$$

if  $n-1$  is linked to  $j$  and  $n$  is linked to  $i$  in  $w$ . This is illustrated graphically in Figure 9.

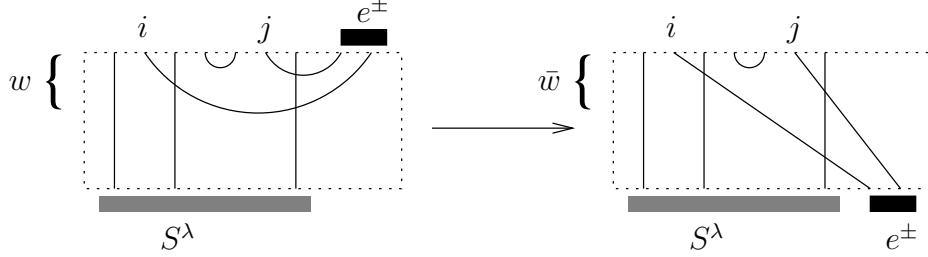


FIGURE 9. An example of the effect of the map  $\phi_2$

Arguing as for  $\phi_1$  we can show that  $\phi_2$  is an isomorphism of vector spaces. Thus we will be done if we can show that  $\phi_2$  commutes with the action of  $B_{n-2}$ .

First consider the action of  $\tau \in \Sigma_{n-2}$ . The actions of  $\phi_2$  and  $\tau$  are illustrated schematically in Figure 10. Again we use the fact that  $\tau(\overline{m}) = \overline{\tau(w)}$ , while in the bottom pair of diagrams we have used the action of  $e^\pm$  on each side, which in each case gives a coefficient of  $\pm 1$ . We see that the actions of  $\phi_2$  does commute with  $\tau$  as required.

It remains to check that  $\phi_2$  commutes with the action of  $X_{kl} \in B_{n-2}$ . If  $\{k, l\}$  is disjoint from  $\{i, j\}$  then it is clear that  $X_{k,l}$  commutes with  $\phi_2$ . If  $k = i$  and  $l = j$  it is easy to verify that

$$X_{ij} e^\pm X_{w,1,id} = 0 \quad \text{and} \quad X_{i,j} \phi_2 (e^\pm X_{w,1,id} \otimes x) = 0.$$

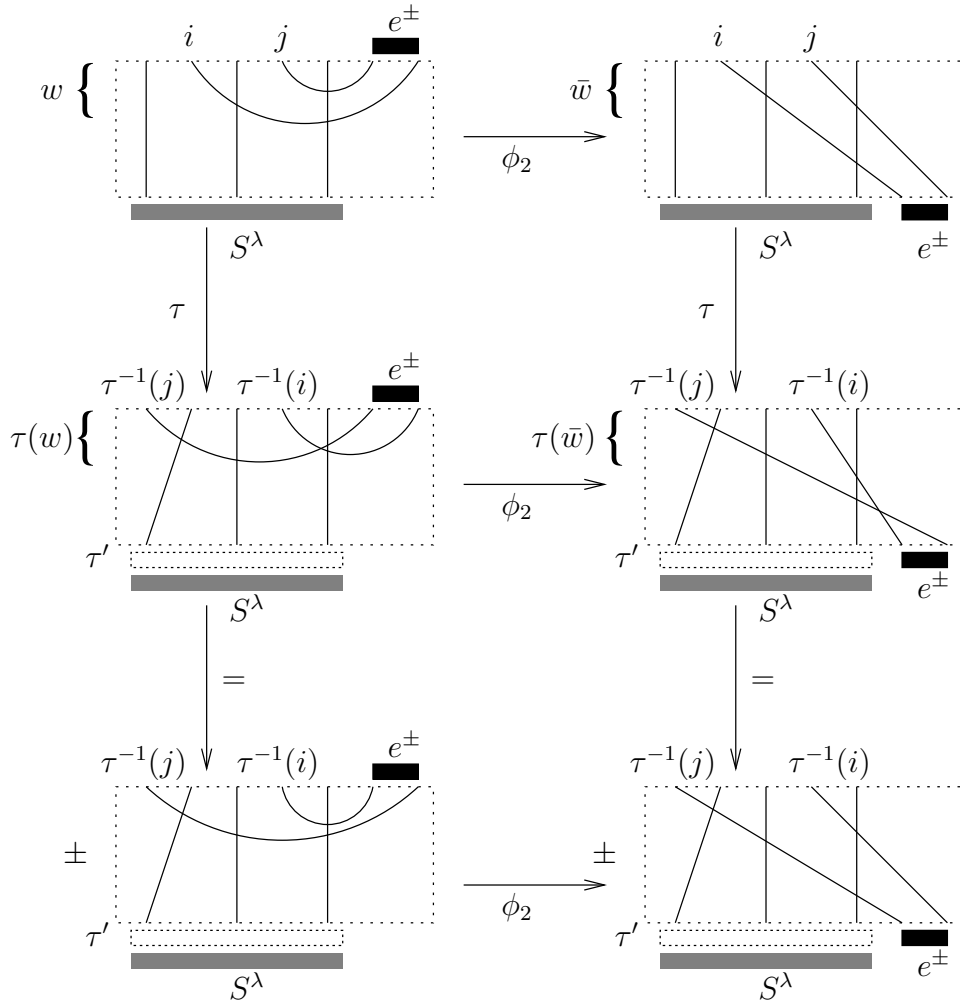
Thus we just have to check what happens when  $k = i$  and  $l \neq j$ . There are two cases: (i)  $l$  is a free node in  $w$ , and (ii)  $l$  is linked to some node  $h$  in  $w$ .

Case (i) is illustrated schematically in Figure 11. The lower left diagram in Figure 11 represents 0 as it lies in  $W_1$ . The lower right diagram represents 0 as there is a decrease in the number of propagating lines. Therefore the dotted arrow is an equality and the diagram commutes.

Case (ii) is illustrated schematically in Figure 12. Again we see that  $X_{il}$  commutes with the action of  $\phi_2$ , and so we are done.  $\square$

## 6. TRANSLATION EQUIVALENCE FOR THE BRAUER ALGEBRA

In Section 4 we saw how translation equivalence of weights for a tower of recollement implies Morita equivalences of the corresponding blocks (when the blocks are truncated to contain the same number of simples). We will now reinterpret this in the language of alcove geometry in the case of the Brauer algebra.

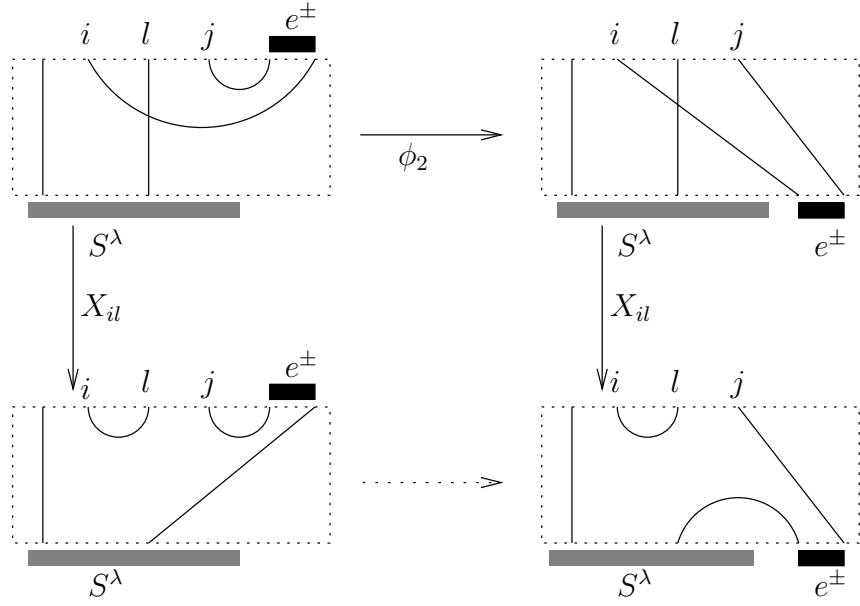
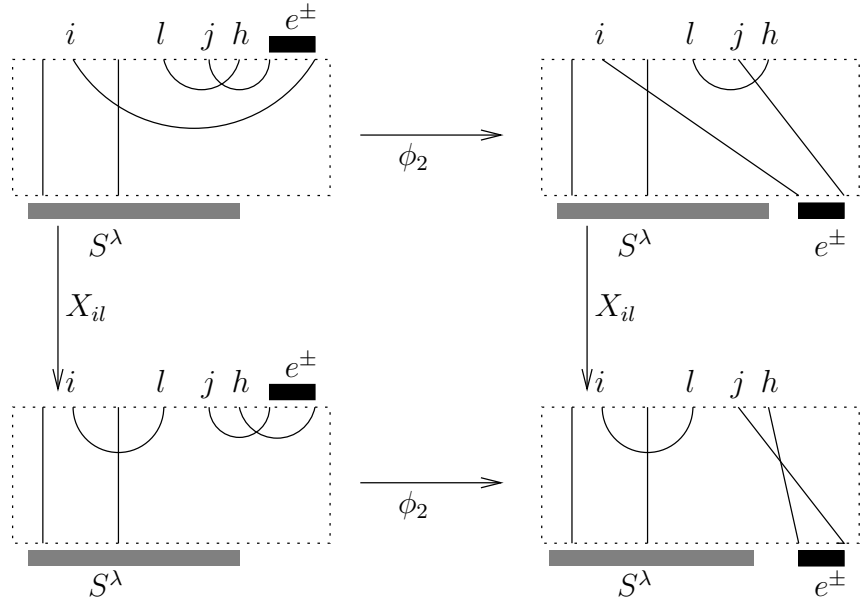
FIGURE 10. A diagrammatic illustration that  $\phi_2\tau = \tau\phi_1$ 

Given a partition  $\lambda$ , we saw in (5) that the set  $\text{supp}(\lambda)$  consists of those partitions obtained from  $\lambda$  by the addition or subtraction of a box from  $\lambda$ , all with multiplicity one. We denote by  $\lambda \pm \epsilon_i$  the composition obtained by adding/subtracting a box from row  $i$  of  $\lambda$ .

**Lemma 6.1.** *If  $\lambda' = \lambda \pm \epsilon_i$  then there cannot exist a reflection hyperplane separating  $\lambda$  from  $\lambda'$ .*

*Proof.* Suppose that  $R$  is a reflection hyperplane between  $\lambda$  and  $\lambda'$ , and denote their respective reflections by  $r(\lambda)$  and  $r(\lambda')$ . Either  $\lambda$  is the reflection of  $\lambda'$  or the line from  $\lambda$  to  $\lambda'$  is not orthogonal to the hyperplane.

The former case is impossible as two weights differing by one box cannot be in the same block. For the latter case, note that the distance between  $\lambda$  and  $\lambda'$  is one. Therefore at least one of the distances from  $\lambda$  to  $r(\lambda)$  and  $\lambda'$  to  $r(\lambda')$  is less than one. But this is impossible, as  $r(\lambda)$  and  $r(\lambda')$  are also elements of the lattice of weights.  $\square$

FIGURE 11. The action of  $X_{il}$  and  $\phi_2$ : case (i)FIGURE 12. The action of  $X_{il}$  and  $\phi_2$ : case (ii)

Given a facet  $F$ , we denote by  $\overline{F}$  the closure of  $F$  in  $\mathbb{E}^\infty$ . This will consist of a union of facets.

**Lemma 6.2.** *If  $\lambda' \in \text{supp}(\lambda)$  and  $\lambda' \in F$  for some facet  $F$  then*

$$|\mathcal{B}(\lambda') \cap \text{supp}(\lambda)| > 1$$

if and only if  $\lambda \in \overline{F} \setminus F$ .

*Proof.* We first show that if  $|\mathcal{B}(\lambda') \cap \text{supp}(\lambda)| > 1$  then  $\lambda \in \overline{F} \setminus F$ . By the interpretation of blocks in terms of contents of partitions (Theorem 2.1) there is precisely one other weight  $\lambda''$  in  $\text{supp}(\lambda)$  in the same block as  $\lambda'$ . Also, one of these weights is obtained from  $\lambda$  by adding a box, and one by subtracting a box. But this implies that  $\lambda'$  is the reflection of  $\lambda''$  about some hyperplane  $H$ ; this reflection must fix the midpoint on the line from  $\lambda'$  to  $\lambda''$ , which is  $\lambda$ , and so  $\lambda \in H$ .

By Lemma 6.1 there is no hyperplane separating  $\lambda$  from  $\lambda'$ . However to complete the first part of the proof we still need to show that if  $\lambda' \in H'$  for some hyperplane  $H'$  then  $\lambda \in H'$  too. First note that by Proposition 3.2(ii)  $H$  and  $H'$  must be  $(i, j)_-$  and  $(k, l)_-$  hyperplanes respectively for some quadruple  $i, j, k, l$ . It is easy to check that either  $(i, j)_-$  fixes  $H'$  or  $(i, j)_- \cdot_\delta H'$  is an  $(i, l)$ -hyperplane. But  $\lambda'' \in (i, j)_- \cdot_\delta H'$  is dominant and so  $(i, j)_-$  must fix  $H'$ . Hence  $\lambda'' \in H'$  and as  $\lambda$  is the midpoint between  $\lambda'$  and  $\lambda''$  we must have  $\lambda \in H'$ .

For the reverse implication, suppose that  $\lambda \in \overline{F} \setminus F$ . Then for all hyperplanes  $H'$  with  $\lambda' \in H'$  we have  $\lambda \in H'$  and there is (at least) one hyperplane  $H$  with  $\lambda \in H$  and  $\lambda' \notin H$ . Suppose that  $H$  is an  $(i, j)_-$ -hyperplane, and consider  $\lambda'' = (i, j)_- \cdot_\delta \lambda'$ .

If  $\lambda'' \in X^+$  then we are done. Otherwise by Lemma 6.1 we have that  $\lambda''$  must lie on the boundary of the dominant region, and hence in some  $(k, l)$ -hyperplane  $\tilde{H}$ . Now  $(i, j)_- \cdot_\delta \lambda'' = \lambda'$  and hence  $\lambda' \in (i, j)_- \cdot_\delta \tilde{H} = H' \neq \tilde{H}$  (as  $\lambda' \in X^+$ ). Therefore we must have  $\lambda \in H'$ . But  $\lambda$  is fixed by  $(i, j)_-$  and so  $\lambda \in \tilde{H} \cap H'$ . This implies that  $\lambda \notin X^+$  which is a contradiction. Thus we have shown that  $\lambda'' \in X^+$  and so  $|\mathcal{B}(\lambda') \cap \text{supp}(\lambda)| > 1$ .  $\square$

**Theorem 6.3.** *If  $\lambda$  is in an alcove then  $\mu$  is in the same translation class as  $\lambda$  if and only if it is in the same alcove.*

*Proof.* By Lemmas 6.1 and 6.2 it is enough to show that if  $\mu$  is in the same alcove as  $\lambda$  then  $\mu$  can be obtained from  $\lambda$  by repeatedly adding or subtracting a box without ever leaving this alcove.

Suppose that  $\lambda$  and  $\mu$  are in the same alcove, and set  $x = \lambda + \rho_\delta$  and  $y = \mu + \rho_\delta$ , the corresponding vectors in  $A^+$ . Recall that there is a permutation  $\pi$  defining the alcove  $A$  introduced in Section 3. We may assume that  $|x_{\pi(1)}| \leq |y_{\pi(1)}|$ . Consider the sequence obtained by repeatedly adding (or subtracting) 1 from  $y_{\pi(1)}$  until we obtain  $x_{\pi(1)}$ . At each stage the vector  $v$  obtained is of the form  $\tau + \rho_\delta$  for some weight  $\tau$ , and the sequence of weights thus obtained are such that each consecutive pair are translation equivalent. Now we repeat the process to convert  $y_{\pi(2)}$  into  $x_{\pi(2)}$  (note that  $y_{\pi(2)}$  and  $x_{\pi(2)}$  have the same sign, and so the chain of weights constructed will always have  $\pi(2)$ -coordinate satisfying the defining conditions for the alcove). We continue in this manner until we have converted  $y$  into  $x$ . This constructs a chain of translation equivalent weights connecting  $\lambda$  and  $\mu$  and so we are done.  $\square$

**Remark 6.4.** Theorem 6.3 shows that the geometry on the weight space for  $B_n(\delta)$  comes naturally from the induction and restriction functors when the alcoves are non-empty (i.e. for  $\delta > 0$ ).

We would like to extend Theorem 6.3 to the case of two weights in the same facet. However, not all weights in the same facet are in the same translation class. To see this, note that a hyperplane is defined by the equation  $x_i = -x_j$  for some fixed pair  $i$  and  $j$ . Any modification of a weight in such a hyperplane by adding or subtracting a single box cannot alter the value of the  $i$ th or  $j$ th coordinate without leaving the hyperplane. However, we will see that if we also use the modified translation functors introduced in Section 5 then we do get the desired equivalences within facets.

Let  $\text{supp}^2(\lambda) = \text{supp}(\text{supp}(\lambda))$ . This set consists of those partitions obtained from  $\lambda$  by adding two boxes, removing two boxes, or adding a box and removing a box.

**Lemma 6.5.** *Suppose that  $\lambda, \tilde{\lambda} \in X^+$  with  $\tilde{\lambda} \in \text{supp}^2(\lambda)$ . If  $\lambda$  and  $\tilde{\lambda}$  are in the same facet then*

$$\text{supp}^2(\lambda) \cap \mathcal{B}(\tilde{\lambda}) = \{\tilde{\lambda}\}.$$

*Proof.* We take  $\lambda' \in \text{supp}^2(\lambda)$  with  $\lambda' \neq \tilde{\lambda}$  and show that the above assumptions imply that  $\lambda' \notin \mathcal{B}(\tilde{\lambda})$ . There are six possible cases.

(i) Suppose that  $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$  and  $\lambda' = \lambda - \epsilon_k + \epsilon_l$ . For these two weights to be in the same block the boxes  $\epsilon_i$  and  $\epsilon_l$  must pair up (and so must  $\epsilon_j$  and  $\epsilon_k$ ) in the sense of condition (1) for a balanced partition. This implies that there is a simple reflection  $(i, l)_-$  taking  $\lambda - \epsilon_i$  to  $\lambda + \epsilon_l$ , which fixes  $\lambda$ . Hence  $\lambda$  is in the  $(i, l)_-$ -hyperplane. However,  $\tilde{\lambda}$  is not in this hyperplane, contradicting our assumption that they are in the same facet.

(ii) Suppose that  $\tilde{\lambda} = \lambda + \epsilon_i + \epsilon_j$  and  $\lambda' = \lambda + \epsilon_k + \epsilon_l$ . For these two weights to be in the same block the elements  $\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l$  must all be distinct and  $\epsilon_i$  and  $\epsilon_j$  must pair up (and so must  $\epsilon_k$  and  $\epsilon_l$ ). Thus there is a reflection taking  $\lambda$  to  $\tilde{\lambda}$ , which contradicts our assumption.

(iii) Suppose that  $\tilde{\lambda} = \lambda - \epsilon_i - \epsilon_j$  and  $\lambda' = \lambda - \epsilon_k - \epsilon_l$ . This is similar to (ii).

(iv) Suppose that  $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$  and  $\lambda' = \lambda + \epsilon_k + \epsilon_l$ . For these two weights to be in the same block we must have  $j = l$  (say). But then  $\lambda + \epsilon_k + \epsilon_j$  is the reflection of  $\lambda - \epsilon_i + \epsilon_j$  through the  $(i, k)_-$ -hyperplane, and hence  $\lambda + \epsilon_j$  is in the  $(i, k)_-$ -hyperplane. Therefore  $\lambda$  is also in this hyperplane, but  $\tilde{\lambda}$  is not, which gives a contradiction.

(v) Suppose that  $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$  and  $\lambda' = \lambda - \epsilon_k - \epsilon_l$ . This is similar to (iv).

(vi) Suppose that  $\tilde{\lambda} = \lambda + \epsilon_i + \epsilon_j$  and  $\lambda' = \lambda - \epsilon_k - \epsilon_l$ . First note that  $\epsilon_i$  and  $\epsilon_j$  cannot pair up (as this would imply that  $\lambda$  and  $\tilde{\lambda}$  are not in the same facet). So for these two weights to be in the same block we must have  $\epsilon_i$  pairing up with  $\epsilon_k$  (say) and  $\epsilon_j$  pairing up with  $\epsilon_l$ . But then  $\lambda - \epsilon_k$  is the reflection of  $\lambda + \epsilon_i$  through the  $(i, k)_-$ -hyperplane, which implies that  $\lambda$  is in this hyperplane but  $\tilde{\lambda}$  is not, which gives a contradiction.  $\square$

**Lemma 6.6.** *Suppose that  $\lambda, \tilde{\lambda} \in X^+$  with  $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j \in \text{supp}^2(\lambda)$ , and that  $\lambda$  lies on the  $(ij)_-$ -hyperplane. Then  $\lambda$  and  $\tilde{\lambda}$  are in the same facet. Moreover, if  $\mu = w \cdot_\delta \lambda \in X^+$  for some  $w \in W$  then  $\tilde{\mu} = w \cdot_\delta \tilde{\lambda}$  satisfies*

$$\tilde{\mu} = \mu - \epsilon_s + \epsilon_t$$

for some  $s, t$ .

*Proof.* The fact that  $\lambda$  and  $\tilde{\lambda}$  are in the same facet is clear. Now suppose that  $\mu = w \cdot_{\delta} \lambda$  and  $\tilde{\mu} = w \cdot_{\delta} \tilde{\lambda}$ . Then  $\tilde{\mu} = \mu + \beta$  where  $\beta = \pm(\epsilon_s + \epsilon_t)$  or  $\beta = \epsilon_s - \epsilon_t$  for some  $s, t$ . Suppose for a contradiction that  $\beta = \pm(\epsilon_s + \epsilon_t)$ . Note that  $\mu$  and  $\tilde{\mu}$  are in the same facet, and so for any  $(k, l)$ -hyperplane on which  $\mu$  and  $\tilde{\mu}$  lie we must have that  $\beta = \tilde{\mu} - \mu$  lies on the *unshifted*  $(k, l)$ -hyperplane. This implies that  $s \neq k, l$  and  $t \neq k, l$ .

We have a sequence of dominant weights  $\mu, \mu' = \mu \pm \epsilon_s$  and  $\tilde{\mu} = \mu \pm (\epsilon_s + \epsilon_t)$  which are each at distance 1 from their neighbours in the sequence. We have already seen that they all lie on the same set of hyperplanes. Moreover, by Lemma 6.1 there cannot exist a hyperplane separating  $\mu$  from  $\mu'$  or  $\mu'$  from  $\tilde{\mu}$ . So  $\mu, \mu'$  and  $\tilde{\mu}$  all lie in the same facet.

Now consider the image of these three weights under  $w^{-1}$ . We get a corresponding sequence  $\lambda, \lambda'$  and  $\tilde{\lambda}$ . These weights must also lie in a common facet (and hence  $\lambda'$  is dominant) and are distance 1 from their neighbours. This forces  $\lambda' = \lambda - \epsilon_i$  or  $\lambda' = \lambda + \epsilon_j$ . However  $\lambda'$  cannot be in the same facet as  $\lambda$  as it does not lie on the  $(i, j)$ -hyperplane, which gives the desired contradiction.  $\square$

Let  $\text{res}_n^{\lambda, \pm} = \text{pr}^{\lambda} \text{res}_n^{\pm}$  and  $\text{ind}_n^{\lambda, \pm} = \text{pr}^{\lambda} \text{ind}_n^{\pm}$ . We say that  $\lambda$  and  $\mu$  are in the same  $(\pm)$ -translation class if there is a chain of dominant weights

$$\lambda = \lambda^0, \lambda^1, \dots, \lambda^r = \mu$$

such that either  $\lambda^{i+1} \in \text{supp}(\lambda^i)$  with  $\lambda^i$  and  $\lambda^{i+1}$  translation equivalent or  $\lambda^{i+1} \in \text{supp}^2(\lambda^i)$  with  $\lambda^{i+1} = \lambda^i + \epsilon_s - \epsilon_t$  (for some  $s$  and  $t$ ) and  $\lambda^i$  and  $\lambda^{i+1}$  are  $(\text{res}^{\lambda^i, \pm}, \text{ind}^{\lambda^{i+1}, \pm})$ -translation equivalent.

Suppose that  $\lambda, \tilde{\lambda} \in \Lambda_n$  with  $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$ , and that  $\lambda$  lies in the  $(i, j)$ -hyperplane. By Lemmas 6.5 and 6.6 we have a bijection  $\theta : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\tilde{\lambda})$  which restricts to a bijection  $\theta : \mathcal{B}_n(\lambda) \rightarrow \mathcal{B}_n(\tilde{\lambda})$ . By Corollary 5.3, Theorem 5.5, Lemmas 6.5 and 6.6, and standard properties of  $\text{ind}$  and  $\text{res}$  it is clear that weights  $\lambda$  and  $\tilde{\lambda}$  are  $(\text{res}^{\lambda, \pm}, \text{ind}^{\tilde{\lambda}, \pm})$  translation equivalent. It is also easy to see that the adjointness isomorphism is multiplicative. Thus we can apply Theorem 4.10 and get a Morita equivalence between the two blocks  $B_n(\lambda)$  and  $B_n(\tilde{\lambda})$ .

**Theorem 6.7.** *If  $\lambda$  and  $\mu$  are in the same facet then they are in the same  $(\pm)$ -translation class.*

*Proof.* By Lemmas 6.2 and 6.5 it is enough to show that if  $\lambda$  and  $\mu$  are in the same facet then there is a chain of dominant weights

$$\lambda = \lambda^0, \lambda^1, \dots, \lambda^r = \mu$$

in the same facet such that  $\lambda^{i+1} \in \text{supp}(\lambda^i)$  or  $\lambda^{i+1} \in \text{supp}^2(\lambda^i)$  for each  $i$ .

Let  $x = \lambda + \rho_{\delta}$  and  $y = \mu + \rho_{\delta}$ , and recall that in Section 3 we associated a function  $f$  to each facet (rather than just a permutation  $\pi$  as for an alcove). The proof now proceeds exactly as for the alcove case (Theorem 6.3) replacing  $\pi$  by  $f$ , until we reach some point where  $f(i) = (k, l)$ . In this case we repeatedly add (or subtract) a box from  $y_k$  and subtract (or add) a box to  $y_l$  until we reach  $x_k$  and  $x_l$ . In each of these steps we obtain some  $\lambda^{i+1} = \lambda^i \pm (\epsilon_k - \epsilon_l) \in \text{supp}^2(\lambda)$ .  $\square$

Applying the results on translation and  $(R, I)$ -translation equivalence from Section 4 we deduce

**Corollary 6.8.** *If  $\lambda$  and  $\lambda'$  are in the same facet and  $\mu$  and  $\mu'$  are such that  $\lambda, \mu \in \Lambda_n$ , and  $\lambda', \mu' \in \Lambda_m$ , and  $\mu'$  is the unique weight in  $\mathcal{B}(\lambda')$  in the same facet as  $\mu$ , then we have:*

(i)

$$[\Delta_n(\lambda) : L_n(\mu)] = [\Delta_m(\lambda') : L_m(\mu')]$$

(ii)

$$\text{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \cong \text{Hom}_m(\Delta_m(\lambda'), \Delta_m(\mu'))$$

(iii)

$$\text{Ext}_n^i(\Delta_n(\lambda), \Delta_n(\mu)) \cong \text{Ext}_m^i(\Delta_m(\lambda'), \Delta_m(\mu'))$$

for all  $i \geq 1$ . If further  $\mathcal{B}_n(\lambda)$  and  $\mathcal{B}_m(\lambda')$  contain the same number of simples then the corresponding blocks are Morita equivalent.

By [DWH99, Theorem 3.4] there are always enough local homomorphisms for the Brauer algebra. Further, by Lemma 6.2 any weight that is not adjacent to a weight in a less singular facet is translation equivalent to a weight of smaller total degree. Thus if  $\delta > 0$  then every weight can be reduced to a weight in some alcove by translation equivalence and repeated applications of Proposition 4.9. This implies

**Corollary 6.9.** *If  $\delta > 0$  then the decomposition numbers  $[\Delta_n(\lambda) : L_n(\mu)]$  for arbitrary  $\lambda$  and  $\mu$  are determined by those for  $\lambda$  and  $\mu$  in an alcove.*

Note that the restriction on  $\delta$  is necessary, as for  $\delta < 0$  there are no weights in an alcove. In fact for  $\delta = -2m$  or  $\delta = -2m + 1$  any dominant weight is  $\delta$ -singular of degree at least  $m$ . For the rest of this section we will see what more can be said in such cases.

We will denote the set of all partitions  $\lambda$  with at most  $m$  non-zero parts by  $\Lambda^{\leq m}$ , and the set of those with at most  $m + 1$  non-zero parts with  $\lambda_{m+1} \leq 1$  by  $\Lambda^{\leq m, 1}$ . (Note that  $\Lambda^{\leq m}$  is precisely the set of weights considered in Theorem 3.10.) Such weights lie in a union of facets, but we shall see that together they play a role analogous to that played by the fundamental alcove in the  $\delta > 0$  case.

We begin by noting

**Proposition 6.10.** (i) *For  $\delta = -2m$ , every  $\delta$ -singular weight of degree  $m$  is in the same block as a unique element of  $\Lambda^{\leq m}$ .*

(ii) *For  $\delta = -2m + 1$ , every  $\delta$ -singular weight of degree  $m$  is in the same block as a unique element of  $\Lambda^{\leq m, 1}$ .*

*Proof.* (i) Suppose that  $\delta = -2m$  and let  $\lambda$  be a  $\delta$ -singular weight of degree  $m$ . Then  $\lambda + \rho_\delta$  is of the form

$$(\dots, x_1, \dots, x_2, \dots, x_m, \dots, (0), \dots, -x_m, \dots, -x_2, \dots, -x_1, \dots, -n, -(n+1), \dots)$$

where the only elements of equal modulus are those of the form  $\pm x_i$ , and the bracketed 0 may or may not appear. Note that as  $\lambda$  is a finite weight the tail of  $\lambda + \rho_\delta$  will equal the tail of  $\rho_\delta$ , i.e. has value  $-n$  in position  $m + 1 + n$  for all  $n \gg 0$ , and we assume that this holds for the  $n$  in the expression above (and similar expressions to follow).

First suppose that  $\lambda + \rho_\delta$  contains 0. Then  $\lambda + \rho_\delta$  is in the same  $W_a$ -orbit as

$$\mu + \rho_\delta = (x_1, x_2, \dots, x_m, 0, \dots, -x_m, \dots, -x_2, \dots, -x_1, \dots, -n, -(n+1), \dots).$$

Thus the  $n-1$  coordinates between the entries 0 and  $-n$  must be strictly decreasing, which forces

$$\mu + \rho_\delta = (x_1, x_2, \dots, x_m, 0, -1, -2, -3, \dots).$$

Hence we deduce that  $\mu \in \Lambda^{\leq m}$  as required.

Next suppose that  $\lambda + \rho_\delta$  does not contain 0. Then there are two cases depending on the parity of the number of positive entries in  $\lambda + \rho_\delta$ . The first case is when  $\lambda + \rho_\delta$  is in the same  $W_a$ -orbit as

$$\mu + \rho_\delta = (x_1, x_2, \dots, y, \dots, x_m, \dots, -x_m, \dots, -x_2, \dots, -x_1, \dots, -n, -(n+1), \dots)$$

where  $y$  is some positive integer and all entries after  $x_m$  are negative. Arguing as above we see that

$$\mu + \rho_\delta = (x_1, x_2, \dots, y, \dots, x_m, -1, -2, -3, \dots).$$

But this vector is  $\delta$ -singular of degree  $m+1$ , which contradicts our assumptions on  $\lambda$ .

The second case is when  $\lambda + \rho_\delta$  is in the same  $W_a$ -orbit as

$$\mu + \rho_\delta = (x_1, x_2, \dots, x_m, \dots, -x_m, \dots, -x_2, \dots, -x_1, \dots, -n, -(n+1), \dots)$$

where all entries after  $x_m$  are negative. But this implies that  $\mu + \rho_\delta$  has  $n$  strictly decreasing coordinates between the entries 0 and  $-n$ , which is impossible.

The argument for (ii) is very similar. We see that  $\lambda + \rho_\delta$  is in the same  $W_a$ -orbit as either

$$\mu + \rho_\delta = (x_1, x_2, \dots, x_m, \dots, -x_m, \dots, -x_2, \dots, -x_1, \dots, -n - \frac{1}{2}, -(n+1) - \frac{1}{2}, \dots)$$

or

$$\mu + \rho_\delta = (x_1, x_2, \dots, y, \dots, x_m, \dots, -x_m, \dots, -x_2, \dots, -x_1, \dots, -n - \frac{1}{2}, -(n+1) - \frac{1}{2}, \dots)$$

where in each case all entries after  $x_m$  are negative.

In the first case we deduce as above that

$$\mu + \rho_\delta = (x_1, x_2, \dots, x_m, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots)$$

and so  $\mu \in \Lambda^{\leq m} \subset \Lambda^{\leq m, 1}$  as required. In the second case, as  $\mu + \rho_\delta$  must be  $\delta$ -singular of degree  $m$ , we deduce that

$$\mu + \rho_\delta = (x_1, x_2, \dots, y, \dots, x_m, -\frac{1}{2}, -\frac{3}{2}, -\dots, -y+1, \hat{y}, -y-1, \dots)$$

where  $\hat{y}$  denotes the omission of the entry  $y$ . But this element is in the same  $W_a$ -orbit as

$$\nu + \rho_\delta = (x_1, x_2, \dots, x_m, \frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots)$$

(by swapping  $y$  and  $-\frac{1}{2}$  with a change of signs, and rearranging to get a decreasing sequence).

Thus  $\lambda$  is in the same  $W_a$ -orbit as  $\nu$ , and  $\nu = (\nu_1, \dots, \nu_m, 1) \in \Lambda^{\leq m, 1}$  as required.  $\square$

Although the weights in  $\Lambda^{\leq m}$  (respectively in  $\Lambda^{\leq m, 1}$ ) lie in several different facets, the next result shows that all these facets have equivalent representation theories.

**Proposition 6.11.** *Let  $\delta = -2m$  (respectively  $\delta = -2m + 1$ ) and  $\lambda \in \Lambda^{\leq m}$  (respectively  $\lambda \in \Lambda^{\leq m,1}$ ). Then  $\lambda$  and  $\lambda'$  are translation equivalent if and only if  $\lambda' \in \Lambda^{\leq m}$  (respectively  $\lambda' \in \Lambda^{\leq m,1}$ ).*

*Proof.* Note that all weights in  $\Lambda^{\leq m}$  (respectively in  $\Lambda^{\leq m,1}$ ) are  $\delta$ -singular of degree  $m$ , and that any pair of such weights can be linked by a chain of weights in the same set differing at each stage only by the addition or subtraction of a single block. By Lemma 6.2 we see that any such pair is translation equivalent.

For the reverse implication, we first consider the  $\delta = -2m$  case, with  $\lambda \in \Lambda^{\leq m}$ , and suppose that  $\lambda' \in \text{supp}(\lambda)$  is not an element of  $\Lambda^{\leq m}$ . Then we must have  $\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_m, 1)$ . Now  $x = \lambda + \rho_\delta$  and  $x' = \lambda' + \rho_\delta$  differ only in the  $m+1$ st coordinate, which is 0 or 1 respectively. If  $f$  is the function associated to the facet containing  $x$  and  $f'$  is the corresponding function for  $x'$  then the only difference is that  $f(1) = m+1$  while  $f'(1) = (m+1, m+2)$ . Thus  $\lambda' \in \overline{F} \setminus F$ , where  $F$  is the facet containing  $\lambda$ , and so by Lemma 6.2 this pair cannot be translation equivalent.

The case  $\delta = -2m + 1$  is similar. Arguing as above we have that  $\lambda' = (\lambda_1, \dots, \lambda_m, 1, 1)$  or  $\lambda' = (\lambda_1, \dots, \lambda_m, 2)$ , and in each case it is easy to show that the pair  $\lambda$  and  $\lambda'$  are not translation equivalent.  $\square$

Combining Propositions 6.10 and 6.11 we deduce that for  $\delta < 0$  and  $\lambda, \lambda' \in \Lambda^{\leq m}$  (respectively  $\lambda, \lambda' \in \Lambda^{\leq m,1}$ ) there is a bijection  $\theta : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda')$  which as before we will denote by  $\theta(\mu) = \mu'$ . Applying the results from Section 4 we obtain

**Corollary 6.12.** *Let  $\delta < 0$  and  $\lambda, \lambda' \in \Lambda^{\leq m}$  (respectively  $\lambda, \lambda' \in \Lambda^{\leq m,1}$ ). If  $\lambda, \lambda' \in \Lambda_n$  and  $\lambda', \mu' \in \Lambda_l$  then (i–iii) of Corollary 6.8 hold. If further  $\mathcal{B}_n(\lambda)$  and  $\mathcal{B}_l(\lambda')$  contain the same number of elements then the corresponding blocks are Morita equivalent.*

As in Corollary 6.9, we obtain the following application of Proposition 4.9.

**Corollary 6.13.** *If  $\delta < 0$  then the decomposition numbers  $[\Delta_n(\lambda) : L_n(\mu)]$  for arbitrary  $\lambda$  and  $\mu$  are determined by those for  $\lambda$  and  $\mu$  in a singular facet of degree  $m$ .*

Combining Corollaries 6.9 and 6.13 with our earlier remarks we obtain

**Theorem 6.14.** *For  $\delta \in \mathbb{Z}$  non-zero the decomposition numbers  $[\Delta_n(\lambda) : L_n(\mu)]$  for arbitrary  $\lambda$  and  $\mu$  are determined by those for  $\lambda$  and  $\mu$  in  $\mathcal{B}(0)$ .*

Thus (at least at the level of decomposition numbers) is it enough to restrict attention to a single block of the Brauer algebra.

**Remark 6.15.** The decomposition numbers for the module  $\Delta_n(0)$  are known by [CDM05, Proposition 5.1 and Theorem 5.2].

We would also like the representation theory to be independent of  $\delta \in \mathbb{Z}$ , in the sense that it should depend only on the geometry of facets. For weights in alcoves, this would in large part follow if we could show that decomposition numbers are given by some kind of parabolic Kazhdan-Lusztig polynomials. In the remaining sections we will consider some evidence for this.

## 7. BLOCK GRAPHS FOR THE BRAUER ALGEBRA

Recall from Section 2 the definition of a maximal balanced partition. Let  $\mathbf{MBS}_\delta(\lambda)$  be the directed graph with vertex set  $V_\delta(\lambda)$  and edge  $\mu \rightarrow \tau$  if  $\mu$  is a maximal  $\delta$ -balanced subpartition of  $\tau$ .

The above graph appears to depend both on  $\lambda$  and  $\delta$ , while the alcove geometry associated to  $W_a$  does not. Let  $\mathbf{Alc}$  be the directed graph with vertex set the set of alcoves for  $W_a$  in  $A^+$ , and an edge  $A \rightarrow B$  if the closures of  $A$  and  $B$  meet in a hyperplane and this hyperplane separates  $A_0$  and  $B$ . (Note that the former condition corresponds to  $B = (ij)_-A$  for some reflection  $(ij)_-$ .)

Our goal in this section is to show that all the graphs  $\mathbf{MBS}_\delta(\lambda)$  are in fact isomorphic, and are isomorphic to the alcove graph  $\mathbf{Alc}$ . This will be our first evidence that the representation theory depends only on the geometry of facets.

Recall that for  $\lambda \in X^+$  we have  $\lambda + \rho_\delta \in A^+$  (the set of strictly decreasing sequences in  $\mathbb{E}^\infty$ ) and the  $\delta$ -dot action of  $W_a$  on  $\lambda$  corresponds to the usual action of  $W_a$  on  $\lambda + \rho_\delta$ . For the rest of this section we will work with the usual action of  $W_a$  on  $A^+$ .

For  $v \in A^+$  we define

$$V(v) = W_a v \cap A^+.$$

We define a partial order on  $A^+$  by setting  $x \leq y$  if  $y - x \in \mathbb{E}^f$  and all entries in  $y - x$  are non-negative. For  $v \in A^+$  we define a directed graph  $\mathbf{G}(v)$  with vertex set  $V(v)$  and arrows given as follows. If  $x, y \in V(v)$ , we set  $x \rightarrow y$  if and only if  $x < y$  and there is no  $z \in V(v)$  with  $x < z < y$ . The reason for introducing this graph is clear from

**Proposition 7.1.** *For  $\lambda \in X^+$  we have  $\mathbf{MBS}_\delta(\lambda^T) \cong \mathbf{G}(\lambda + \rho_\delta)$ .*

*Proof.* By Theorem 3.3 we have a bijection between  $V_\delta(\lambda^T)$  and  $V(\lambda + \rho_\delta)$ . Moreover, for  $\mu^T, \nu^T, \tau^T \in V_\delta(\lambda^T)$  we have  $\mu^T \subset \nu^T \subset \tau^T$  if and only if  $\mu + \rho_\delta < \nu + \rho_\delta < \tau + \rho_\delta$ . Thus the two graph structures on these vertex sets are preserved under the correspondence.  $\square$

Recall the definition of singletons from Section 3. Define  $v_{reg}$  to be the subsequence of  $v$  consisting only of its singletons. For example, if  $v$  begins  $(9, 8, 7, 0, -1, -2, -7, -9, -11, \dots)$  then  $v_{reg}$  begins  $(8, 0, -1, -2, -11, \dots)$ . Note that if  $v \in A^+$  then  $v_{reg} \in A^+$  and  $|(v_{reg})_i| \neq |(v_{reg})_j|$  for all  $i \neq j$ . Therefore  $v_{reg}$  is a regular element in  $\mathbb{E}^\infty$  (as it does not lie on any reflecting hyperplane). We define the *regularisation map*  $\text{Reg} : A^+ \rightarrow A^+$  by setting

$$\text{Reg}(v) = v_{reg}.$$

The key result about the regularisation map is

**Proposition 7.2.** *For all  $v \in A^+$  we have*

$$\mathbf{G}(v) \cong \mathbf{G}(v_{reg}).$$

*Proof.* We first observe that the map  $\text{Reg}$  gives rise to a bijection between  $V(v)$  and  $V(v_{reg})$ . For the set of doubletons is an invariant of the elements in  $V(v)$ , and given this set there is a unique way of adding the doubletons into an element of  $V(v_{reg})$  keeping the sequence strictly decreasing. Now suppose that  $x, y \in A^+$  and  $a \in \mathbb{R}$  are such that

$$s = (x_1, \dots, x_i, a, x_{i+1}, \dots) \in A^+ \quad \text{and} \quad t = (y_1, \dots, y_j, a, y_{j+1}, \dots) \in A^+.$$

Then it is easy to see that  $x < y$  if and only if  $s < t$ . However, this implies that the set of edges coincide under the map  $\text{Reg}$ , as required.  $\square$

**Corollary 7.3.** *For all  $v, v' \in A^+$  we have*

$$\mathbf{G}(v) \cong \mathbf{Alc}.$$

*Hence for all  $\delta, \delta' \in \mathbb{Z}$  and  $\lambda, \lambda' \in X^+$  we have*

$$\mathbf{MBS}_\delta(\lambda) \cong \mathbf{MBS}_{\delta'}(\lambda').$$

*Proof.* Note that any  $v \in \text{Reg}(A^+)$  lies inside an alcove. For any vector  $v \in \text{Reg}(A^+)$  the maximal weights below  $v$  in the same orbit lie in the alcoves below and adjacent to the alcove containing  $v$ . Thus for  $v \in \text{Reg}(A^+)$  it is clear that we have  $\mathbf{G}(v) \cong \mathbf{Alc}$ . Now the result follows for general  $v$  from Proposition 7.1, and in its  $\mathbf{MBS}$  form from Proposition 7.2.  $\square$

It will be convenient to give  $\mathbf{Alc}$  the structure of a graph with coloured edges. An edge in  $\mathbf{Alc}$  corresponds to reflection from an alcove  $A$  to an alcove  $B$  through the facet separating them. The action of  $W_a$  on weights induces a corresponding action on facets, and we shall say that two edges have the same colour if and only if the corresponding facets lie in the same orbit.

We conclude this section with one final graph  $\mathbf{Par}_e^+$  isomorphic to  $\mathbf{Alc}$ , whose structure can be described explicitly.

We fix the element  $v = (-1, -2, -3, -4, \dots) \in A^+$ . Using the action of  $W_a$  we can see that every  $x \in V(v)$  corresponds uniquely to a strictly decreasing partition with an even number of parts, obtained by ignoring all parts of  $x$  which are negative. For example, the element  $x = (6, 5, 3, 1, -2, -4, -7, -8, \dots)$  corresponds to  $(6, 5, 3, 1)$  while  $v$  corresponds to  $\emptyset$ . Thus if we write  $P_e^+$  for the set of strictly decreasing partitions with an even number of parts then we have a bijection

$$\phi : V(v) \longrightarrow P_e^+.$$

Consider the usual partial order  $\subseteq$  on  $P_e^+$  given by inclusion of partitions (viewed as Young diagrams). It is clear that the partial order  $\leq$  on  $V(v)$  corresponds to the partial order  $\subseteq$  on  $P_e^+$  under the bijection  $\phi$ . Define a graph  $\mathbf{Par}_e^+$  with vertex set  $P_e^+$  and an arrow  $\lambda \rightarrow \mu$  if and only if  $\lambda \subset \mu$  and there is no  $\nu \in P_e^+$  with  $\lambda \subset \nu \subset \mu$ . It is easy to verify that the map  $\phi$  induces a graph isomorphism between  $\mathbf{G}(v)$  and  $\mathbf{Par}_e^+$ .

The graph  $\mathbf{Par}_e^+$  can easily be described explicitly as follows. For  $\lambda, \mu \in P_e^+$ , there is an arrow  $\lambda \rightarrow \mu$  if and only if either

$$\lambda = (\lambda_1, \dots, \lambda_n) \quad \text{and} \quad \mu = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_n) \quad (20)$$

or

$$\lambda = (\lambda_1, \dots, \lambda_n) \text{ with } \lambda_n \geq 3 \quad \text{and} \quad \mu = (\lambda_1, \dots, \lambda_n, 2, 1). \quad (21)$$

To see this, first observe that in both cases there is no  $\nu \in P_e^+$  with  $\lambda \subset \nu \subset \mu$ . Moreover, if  $\lambda, \mu \in P_e^+$  with  $\lambda \subset \mu$  then  $\mu$  can be obtained from  $\lambda$  by applying (20) and (21) repeatedly.

## 8. COXETER SYSTEMS AND PARABOLIC KAZHDAN-LUSZTIG POLYNOMIALS

In this section we will introduce parabolic Kazhdan-Lusztig polynomials associated with the pair  $(W_a, W)$ . We briefly review the relevant theory; details can be found in [Hum90] and [Soe97b].

Recall that a Coxeter system is a pair  $(G, S)$  consisting of a group  $G$  and a set  $S$  of generators of  $G$  such that all relations in  $G$  are of the form

$$(ss')^{m(s,s')} = 1$$

where  $m(s, s) = 1$  and  $m(s, s') = m(s', s) \geq 2$  otherwise (including the possibility that  $m(s, s') = \infty$  denoting no relation between  $s$  and  $s'$ ). Note that the group  $G$  does not need to be finite (although this is often assumed). Given a Coxeter system, the associated Coxeter graph is the graph with vertices the elements of  $S$ , and  $m(s, s') - 2$  edges between  $s$  and  $s'$  (or no edges when  $m(s, s') = \infty$ ). For example the  $D_\infty$  Coxeter graph is given by the graph shown in Figure 13.

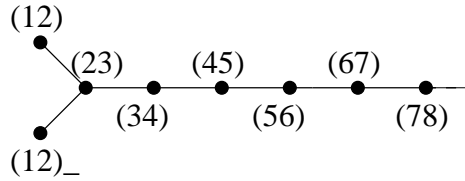


FIGURE 13. The type  $D_\infty$  Coxeter system

It is easy to verify that our group  $W_a$  is generated by the elements  $\{(12)_-, (ii+1) : i \geq 1\}$  and satisfies the relations given by the Coxeter graph in Figure 13. Thus it must be a quotient of the Coxeter group of type  $D_\infty$ . However, for each choice of  $n$ , the subsystem generated by the first  $n$  generators is precisely the type  $D_n$  system (see [Bou68, Planche IV]), and so there can be no further relations, and our group is the type  $D_\infty$  Coxeter group associated to the given generators.

Given a Coxeter system  $(G, S)$ , any subgroup  $G'$  generated by a subset  $S'$  of  $S$  defines a parabolic subsystem  $(G', S')$ . In our case the group  $W$  clearly arises in this way from the generators of the form  $(ii+1)$  and so is a type  $A_\infty$  parabolic subgroup of  $W_a$ .

When  $\delta \geq 0$  there is a bijection from  $W_a$  to the set of alcoves, given by  $w \mapsto w \cdot 0$ . We will henceforth identify elements of  $W_a$  with alcoves via this map. Under this bijection the standard length function on our Coxeter system associated to  $W_a$  (given in terms of the number of terms occurring in a reduced expression for  $w$ ) corresponds to the number of reflection hyperplanes between  $0$  and  $w \cdot_\delta 0$ .

We define  $W^a$  to be the subset of  $W_a$  corresponding to the alcoves in  $X^+$ . By Proposition 3.2(ii) we then have a bijection

$$W \times W^a \rightarrow W_a.$$

We are thus in a position to define  $D_\infty/A_\infty$  parabolic Kazhdan-Lusztig polynomials following [Deo87] (although we use the notation of [Soe97b, Section 3]). Their precise definition and general properties need not concern us, instead we will give a recursive construction

corresponding to stepping away from the root of **Alc**. To do this we will first need to define a partial order on weights (and alcoves).

Two weights  $\lambda$  and  $\mu$  such that  $\mu = w \cdot \lambda$  for some reflection  $w$ , lie in different components of the space formed by removing this hyperplane. We say that  $\lambda < \mu$  if  $\lambda$  is in the component containing the fundamental alcove. This extends to give a partial order on weights, which in turn induces a partial order on alcoves. This agrees with the path-from-root order on **Alc**. Two alcoves are said to be *adjacent* if there is precisely one reflecting hyperplane between them (i.e. they are adjacent in **Alc**).

Suppose that  $\nu$  and  $\mu$  are dominant weights in adjacent alcoves with  $\nu = s \cdot_\delta \mu > \mu$ . Given a dominant weight  $\lambda \in W_a \cdot_\delta \mu$  we define  $\kappa_\lambda(\nu, \mu)$  to be the unique weight such that

$$(\kappa_\lambda(\nu, \mu), \lambda) = (w \cdot_\delta \mu, w \cdot_\delta \nu)$$

i.e.  $(\kappa_\lambda(\nu, \mu), \lambda)$  is an edge of the same colour as  $(\mu, \nu)$  in **Alc**.

We next define certain polynomials  $n_{\nu, \lambda}$  (in an indeterminate  $v$ ) for regular weights  $\lambda$  and  $\mu$  in the following recursive manner. Let  $e_\lambda$  as  $\lambda$  runs over the regular weights be a set of formal symbols.

- (i) We set  $n_{\nu, \lambda} = 0$  if  $\lambda \not\leq \nu$  or  $\lambda \notin W_a \cdot_\delta \nu$  or either  $\lambda$  or  $\nu$  is non-dominant.
- (ii) We set  $n_{0, 0} = 1$  and  $N(0) = e_0$ .
- (iii) For each  $\nu > 0$  regular dominant, there exists some  $\mu$  regular dominant below it such that  $\mu = s \cdot_\delta \nu$  and  $(\nu, \mu)$  are in adjacent alcoves. Pick any such  $\mu$ . Then for any dominant  $\lambda$  with  $\lambda = w \cdot_\delta \nu$  for some  $w$  and  $\kappa = \kappa_\lambda(\nu, \mu)$  we set

$$\hat{n}_{\nu, \lambda} = \text{pr}_+(\kappa) (n_{\mu, \kappa} + v^{l(\kappa) - l(\lambda)} n_{\mu, \lambda})$$

where  $\text{pr}_+(\kappa) = 1$  if  $\kappa \in X^+$  and  $\text{pr}_+(\kappa) = 0$  otherwise. Note that for  $\kappa \in X^+$  we have  $l(\kappa) - l(\lambda) = -1$  if  $\kappa < \lambda$ , respectively  $+1$  if  $\kappa > \lambda$ . Let  $\hat{N}(\nu)$  be the sum

$$\hat{N}(\nu) = \sum_{\lambda} \hat{n}_{\nu, \lambda} e_{\lambda}.$$

and  $R(\nu)$  be the set of  $\lambda < \nu$  such that  $\hat{n}_{\nu, \lambda}(0) \neq 0$ . Then

$$N(\nu) = \hat{N}(\nu) - \sum_{\lambda \in R(\nu)} \hat{n}_{\nu, \lambda}(0) N(\lambda)$$

and  $n_{\nu, \lambda}$  is the coefficient of  $e_\lambda$  in  $N(\nu)$ .

It is a consequence of (Deodhar's generalisation of) Kazhdan-Lusztig theory that this process is well defined (so does not depend on the choice of  $\mu$  in step (iii)), and that each  $n_{\nu, \lambda}$  is a polynomial in  $v$  with  $n_{\nu, \lambda}(0) \neq 0$  only if  $\lambda = \nu$ .

## 9. SOME LOW RANK CALCULATIONS FOR $\delta = 1$

To illustrate the various constructions so far, we will consider the case when  $\delta = 1$ , and examine the regular block containing the weight 0. First we calculate the associated parabolic Kazhdan-Lusztig polynomials, and then we compare these with the representation theoretic results.

We will also need to consider the block containing (1). As this is in the same alcove as 0 these two blocks are translation equivalence. However, in this simple case case we do not obtain any simplification to the calculations by applying the results from Section 6; instead the results can be considered as a verification of the general theory in this special case.

In Figure 14 we have listed all dominant weights of degree at most 16 that are in the same block as the weight 0. We will abbreviate weights in the same manner as partitions (and so write for example  $(1^3)$  instead of  $(1, 1, 1)$ ). An edge between two weights indicates that they are in adjacent alcoves, and the label  $(ij)_-$  corresponds to the reflection hyperplane between them. (Clearly only weights of the form  $(ij)_-$  can arise as such labels.)

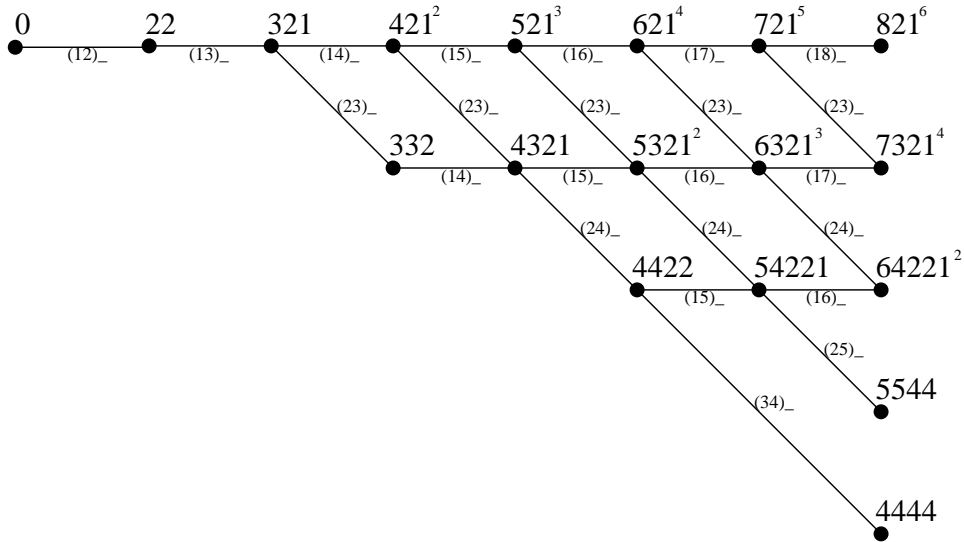


FIGURE 14. The block of the weight 0 for  $\delta = 1$ , up to degree 16

Given this data we can now compute the  $n_{\lambda, \mu}$ . The final results are shown in Figure 15. We start with the weight 0 having  $n_{(0), (0)} = 1$ . Reflecting through  $(12)_-$  we obtain the weight (22), and we see that  $n_{(22), (0)} = v$ . (Note that the term  $n_{\lambda, \lambda}$  is always 1.) Continuing we reflect (22) through  $(13)_-$  to obtain (321). As  $(12)_-(13)_- \cdot_\delta 0 = (2, -1, 3)$  is not dominant we see that the only non-zero term apart from  $n_{(321), (321)}$  is  $n_{(321), (22)} = v$ . Identical arguments give all polynomials  $n_{\nu, \lambda}$  where  $\nu$  is on the top row of Figure 14. For the second row we obtain four terms as  $\nu$  and  $\mu$  both give dominant weights under the action of  $(1i)_-(23)_-$  for suitable  $i$  (as the parallelogram with  $\nu$  as highest term has identically labelled parallel sides).

For (4422) we must observe that

$$(14)_-(23)_-(24)_- \cdot_\delta (4321) = (22)$$

and similar results give the remaining cases. In all of these cases we have no constant terms arising at any stage (apart from in  $n_{\lambda, \lambda}$ ), and hence  $N(\nu) = \hat{N}(\nu)$  for every weight considered.

Next we will determine the structure of certain low rank standard modules for  $B_n(1)$  in the block containing 0. These will then be compared with the Kazhdan-Lusztig polynomials calculated above. We will proceed in stages, and will also need to consider the structure of modules in the block containing (1). The submodule structure of modules will be illustrated

$\mu$	$\lambda$	0	22	321	4211	332	521 <sup>3</sup>	4321	621 <sup>4</sup>	53211	4422	721 <sup>5</sup>	6321 <sup>3</sup>	54221	821 <sup>6</sup>	7321 <sup>4</sup>	64221 <sup>2</sup>	552 <sup>3</sup>	4 <sup>4</sup>
0		1																	
22		$v$	1																
321		.	$v$	1															
4211		.	.	$v$	1														
332		.	.	$v$	.	1													
521 <sup>3</sup>		.	.	.	$v$	.	1												
4321		.	.	$v^2$	$v$	$v$	.	1											
621 <sup>4</sup>		.	.	.	.	.	$v$	.	1										
53211		.	.	.	$v^2$	.	$v$	$v$	.	1									
4422		.	$v^2$	$v$	.	.	.	$v$	.	.	1								
721 <sup>5</sup>		.	.	.	.	.	.	$v$	.	.	.	1							
6321 <sup>3</sup>		.	.	.	.	.	$v^2$	.	$v$	$v$	.	.	1						
54221		.	.	.	.	.	.	$v^2$	.	$v$	$v$	.	.	1					
821 <sup>6</sup>		.	.	.	.	.	.	.	.	.	.	$v$	.	.	1				
7321 <sup>4</sup>		.	.	.	.	.	.	.	$v^2$	.	.	$v$	$v$	.	.	1			
64221 <sup>2</sup>		.	.	.	.	.	.	.	.	$v^2$	.	.	$v$	$v$	.	.	1		
552 <sup>3</sup>		.	.	.	.	$v^2$	.	$v$	.	.	.	.	.	$v$	.	.	.	1	
4 <sup>4</sup>		$v^2$	$v$	.	.	.	.	.	.	.	$v$	.	.	.	.	.	.	.	1

FIGURE 15. The polynomials  $n_{\lambda,\mu}$  for  $\delta = 1$  and  $|\lambda| \leq 16$ 

diagrammatically, where a simple module  $X$  is connected by a line to a simple module  $Y$  above it if there is a non-split extension of  $X$  by  $Y$ . Note that in this section we follow the usual labelling of modules by partitions (as in [CDM05]) and *not* via the transpose map by weights.

9.1. **The case  $n \leq 6$ .** When  $n = 0$  or  $n = 2$  we have

$$\Delta_n(0) = L_n(0)$$

by quasihereditary (and the absence of any other simples in the same block).

When  $n = 4$  we have

$$\Delta_4(22) = L_4(22) \quad \text{and} \quad \Delta_4(0) = \begin{array}{c} L_4(0) \\ | \\ L_4(22) \end{array}$$

by quasi-hereditary and (2).

When  $n = 6$  we have

$$\Delta_6(321) = L_6(321) \quad \text{and} \quad \Delta_6(22) = \begin{array}{c} L_6(22) \\ | \\ L_6(321) \end{array}$$

as in the case  $n = 4$ . For the remaining module  $\Delta_6(0)$  we know that  $[\Delta_6(0) : L_6(22)] = 1$  by localising to  $n = 4$ . Applying Proposition 2.2 with  $\mu = (32)$  (as this weight is minimal in its

$$\Delta_6(0) = \begin{array}{c} L_6(0) \\ | \\ L_6(22). \end{array}$$
$$\Delta_1(1) = L_1(1), \quad \Delta_3(21) = L_3(21), \quad \Delta_3(1) = \begin{array}{c} L_3(1) \\ | \\ L_6(21) \end{array}$$
$$\Delta_5(311) = L_5(311), \quad \Delta_5(21) = \begin{array}{c} L_5(21) \\ | \\ L_5(311) \end{array}, \quad \Delta_5(1) = \begin{array}{c} L_5(1) \\ | \\ L_5(21) \end{array}.$$
$$\Delta_7(41^3) = L_7(41^3), \quad \text{and} \quad \Delta_7(311) = \begin{array}{c} L_7(311) \\ | \\ L_5(41^3). \end{array}$$
$$\begin{array}{ccc} \Delta_7(21) = & L_7(21) & \text{and} \quad \Delta_7(1) = L_7(1) \\ & | & | \\ & L_8(311) & L_7(21). \end{array}$$
$$\Delta_8(421^2) = L_8(421^2), \quad \Delta_8(332) = L_8(332), \quad \text{and} \quad \Delta_8(321) = \begin{array}{cc} & L_8(321) \\ & \swarrow \quad \searrow \\ L_8(421^2) & L_8(332). \end{array}$$
$$\begin{array}{ccc} \Delta_8(22) = & L_8(22) & \text{and} \quad \Delta_8(0) = L_8(0) \\ & | & | \\ & L_8(321) & L_8(22). \end{array}$$

**9.4. The case  $n = 9$ .** In this case we have 6 standard modules, labelled by  $(1)$ ,  $(21)$ ,  $(311)$ ,  $(41^3)$ ,  $(51^4)$  and  $(3^3)$ . By (2) there is a homomorphism from each of these standards to the preceding one in the list, except in the case of  $(3^3)$ . For this weight we instead use (3) which tells us that

$$[\Delta_9(311) : L_9(3^3)] = 1.$$

As in earlier cases, we have that

$$\Delta_9(51^4) = L_9(51^4), \quad \Delta_9(3^3) = L_9(3^3), \quad \text{and} \quad \Delta_9(41^3) = L_9(41^3) \\ \downarrow \\ L_9(3^3).$$

The module  $L_9(51^4)$  cannot occur in any other standards, by applying Proposition 2.2 with  $\mu = (41^4)$ , and similarly  $L_9(3^3)$  can only occur in  $\Delta_9(311)$ , by taking  $\mu = (331)$ . By the above observations and localisation to  $n = 7$  we deduce that

$$\Delta_9(311) = \begin{array}{c} L_9(311) \\ \swarrow \quad \searrow \\ L_9(41^3) \quad L_9(3^3) \end{array} \quad \Delta_9(21) = \begin{array}{c} L_9(21) \\ \downarrow \\ L_9(311) \end{array} \quad \Delta_9(1) = \begin{array}{c} L_9(1) \\ \downarrow \\ L_9(21). \end{array}$$

We will need to consider  $\text{res}_{10} \Delta_{10}(321)$ . For this we need to understand the various standard modules arising in the short exact sequence (5) in this case. First note that  $(32)$  and  $(221)$  are the unique weights in their respective blocks when  $n = 9$ . For the weights  $(331)$ ,  $(322)$  and  $(3211)$  there is exactly one larger weight in the same block in each case, respectively  $(4311)$ ,  $(4221)$ , and  $(3321)$ .

It follows from the above remarks, (2), and (5) that  $\text{res}_{10} \Delta_{10}(321)$  has a short exact sequence

$$0 \longrightarrow A \longrightarrow \text{res}_{10} \Delta_{10}(321) \longrightarrow B \longrightarrow 0 \quad (22)$$

where

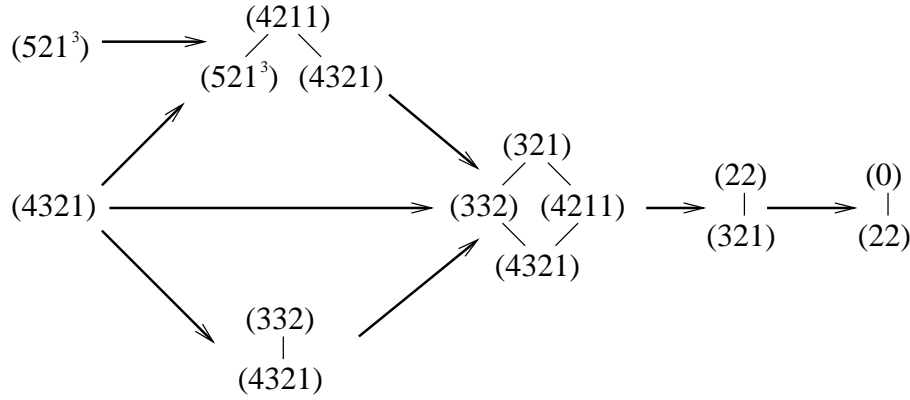
$$A \cong L_9(221) \oplus \begin{array}{c} L_9(311) \\ L_9(41^3) \end{array} \oplus L_9(32) \quad (23)$$

and

$$B \cong L_9(421) \oplus L_9(331) \oplus L_9(322) \oplus L_9(3211) \\ L_9(432) \quad L_9(4311) \quad L_9(4221) \quad L_9(3321). \quad (24)$$

**9.5. The case  $n = 10$ .** From now on, we will summarise the results obtained for each value of  $n$  in a single diagram, together with an explanation of how they were derived. In each such diagram we shall illustrate the structure of individual modules as above, but label simple factors just by the corresponding partition. We will indicate the existence of a homomorphism between two modules by an arrow. (It will be clear which standard module is which by the label of the simple in the head.)

For  $n = 10$  we claim that the structure of the block containing  $(0)$  is given by the data in Figure 16. The structure of the modules  $\Delta_{10}(521^3)$ ,  $\Delta_{10}(4321)$  and  $\Delta_{10}(4211)$ , follows exactly as in the preceding cases for partitions of  $n$  and  $n - 2$ . For  $\Delta_{10}(332)$  we also need to note that  $(332) \not\subset (521^3)$ , and so  $L_{10}(521^3)$  cannot occur.

FIGURE 16. The block containing 0 when  $n = 10$ 

To see that  $L_{10}(521^3)$  cannot occur anywhere else it is enough to note (by Proposition 2.2) that  $\Delta_9(421^3)$  is projective. Similarly  $L_{10}(4321)$  cannot occur in the standards  $\Delta_{10}(22)$  and  $\Delta_{10}(0)$  as  $\Delta_8(431)$  is projective. The structure of  $\Delta_{10}(0)$  and  $\Delta_{10}(22)$  then follows by localisation to  $n = 8$ .

The only remaining module is  $\Delta_{10}(321)$ . It is clear that this must have at most the four factors shown. The multiplicities of  $L_{10}(4211)$  and  $L_{10}(332)$  must be 1 by localisation to the case  $n = 8$ . It remains to show that the final factor has multiplicity 1, that there is a map to the module from  $\Delta_{10}(4321)$ , and that the module structure is as shown.

Consider  $\text{res}_{10} L_{10}(4321) = \text{res}_{10} \Delta_{10}(4321)$ . By (5), and the simplicity of standard modules  $\Delta_n(\lambda)$  when  $\lambda \vdash n$ , this has simple factors

$$L_9(432) \quad L_9(4311) \quad L_9(4221) \quad L_9(3321). \quad (25)$$

If we consider  $\text{res}_{10} \Delta_{10}(4211)$  and  $\text{res}_{10} \Delta_{10}(332)$ , using the structure of  $\Delta_{10}(4211)$  and  $\Delta_{10}(332)$  given above, it is easy to show that neither  $\text{res}_{10} L_{10}(4211)$  nor  $\text{res}_{10} L_{10}(332)$  contain any of the factors in (25). Comparing with (22), (23), and (24) we see that

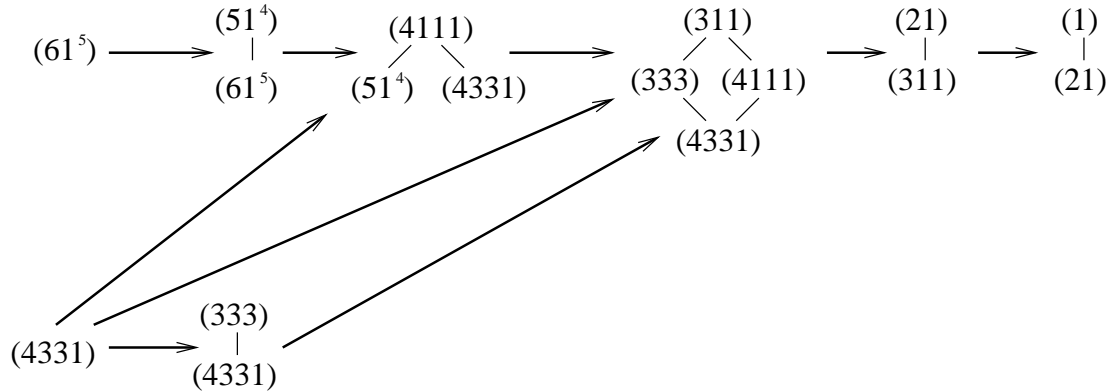
$$[\Delta_{10}(321) : L(4321)] \leq 1.$$

Further, either this simple does occur, or the simples in (25) all occur in  $\text{res}_{10} L_{10}(321)$ .

By (6) we have that  $\text{res}_{10} L_{10}(4211)$  contains  $L_9(421)$  and  $\text{res}_{10} L_{10}(332)$  contains  $L_9(331)$ . (In fact, all factors can be easily determined.) Comparing with (24) we see that both  $L_{10}(4211)$  and  $L_{10}(331)$  must occur above the simple whose restrictions contribute the terms in (25). Thus  $L_{10}(4321)$  must occur and forms the socle of  $\Delta_{10}(321)$ . This completes our verification of the various claims above, and so the block structure is as shown in Figure 16.

**9.6. The case  $n = 11$ .** We claim that the structure of the block containing (1) is given by the data in Figure 17. The structure of the modules  $\Delta_{11}(61^5)$ ,  $\Delta_{11}(4331)$ ,  $\Delta_{11}(51^4)$  and  $\Delta_{11}(333)$  follows as for  $n = 10$  for partitions of  $n$  and  $n - 2$ .

The modules  $\Delta_{10}(51^5)$  and  $\Delta_8(431)$  are projective. Therefore by Proposition 2.2 the simple  $L_{11}(61^5)$  cannot occur in any of the remaining standards, and  $L_{11}(4331)$  cannot occur in  $\Delta_{11}(21)$  or  $\Delta_{11}(1)$ . The structure of these latter two modules now follows by localisation

FIGURE 17. The block containing (1) when  $n = 11$ 

to the case  $n = 9$ . To see that the structure of  $\Delta_{11}(41^3)$  is as illustrated follows from (3) and localisation.

The only remaining module is  $\Delta_{11}(311)$ . By localising this contains  $L_{11}(31^2)$ ,  $L_{11}(3^3)$  and  $L_{11}(41^3)$ , all with multiplicity one. We have eliminated all other possible factors except  $L_{11}(4331)$ . We proceed as for the module  $\Delta_{10}(321)$  above. Restriction of  $\Delta_{11}(311)$  contains  $L_{10}(433)$  with multiplicity one, and this can only arise in the restriction of  $L_{11}(4311)$ . Arguing as in the  $n = 10$  case we also see that  $L_{11}(4311)$  must coincide with the socle of  $\Delta_{11}(311)$ , and so we are done.

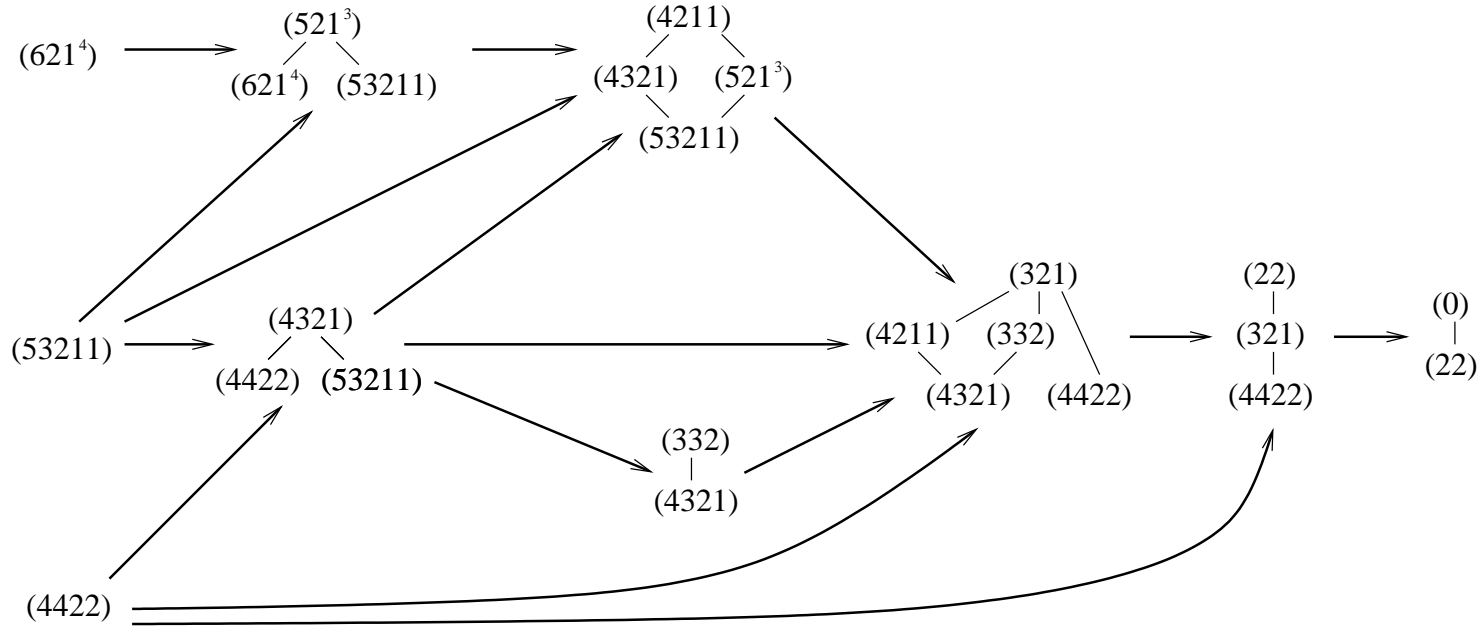
**9.7. The case  $n = 12$ .** We claim that the structure of the block containing (0) is given by the data in Figure 18. As usual, the structure of the modules labelled by partitions of  $n$  and  $n - 2$  is straightforward. The simple  $L_{12}(621^4)$  cannot occur anywhere else as  $\Delta_{11}(521^4)$  is projective.

The modules  $\Delta_{10}(42211)$  and  $\Delta_8(2^4)$  are both projective. Therefore  $L_{12}(53211)$  cannot occur in any standard labelled by a partition of 6 or smaller, while  $L_{12}(4422)$  cannot occur in  $\Delta_{12}(0)$ . The structure of  $\Delta_{12}(0)$  is then clear by localisation.

Next consider  $\Delta_{12}(332)$ . This cannot contain  $L_{12}(53211)$  as  $(332) \not\subset (53211)$ , so it is enough by localisation to verify that  $L_{12}(4422)$  cannot occur. But this is clear, as the restriction of this simple contains  $L_{11}(4421)$ , which is not in the same block as any of the standard modules in the restriction of  $\Delta_{12}(332)$ .

It remains to determine the structure of  $\Delta_{12}(321)$  and  $\Delta_{12}(22)$ . In each case we know the multiplicity of all composition factors by localisation and the remarks above, except for  $L_{12}(4422)$ . Using Proposition 2.4 we see that we have a non-zero homomorphism from  $\Delta_{12}(4422)$  into each of the two standards, and so the multiplicity of  $L_{12}(4422)$  in each case is at least one. The restriction of each of these standards contains precisely one copy of  $L_{11}(4322)$  and  $L_{11}(4421)$ , and so  $L_{12}(4422)$  must occur with multiplicity one in each standard.

To determine the location of  $L_{12}(4422)$  in  $\Delta_{12}(321)$  note that it cannot occur below any composition factor other than  $L_{12}(321)$ , as this would contradict the existence of homomorphisms from  $\Delta_{12}(4211)$  and  $\Delta_{12}(332)$  into  $\Delta_{12}(321)$ . Therefore the structure of this module

FIGURE 18. The block containing 0 when  $n = 12$

must be as shown. For  $\Delta_{12}(22)$ , we have that the simple  $L_{11}(32)$  in the restriction of  $L_{12}(321)$  occurs above  $L_{12}(4421)$ , and hence the structure of  $\Delta_{12}(22)$  must be as shown.

**9.8. A comparison with Kazhdan-Lusztig polynomials.** Suppose that  $W$  is a Weyl group, with associated affine Weyl group  $W_p$ . Soergel has shown [Soe97b, Soe97a] that (provided  $p$  is not too small) the value of the parabolic Kazhdan-Lusztig polynomials  $n_{\lambda\mu}$  (evaluated at  $v = 1$ ) associated to  $W \subset W_p$  determine the multiplicity of the standard module  $\Delta_q(\lambda)$  in the indecomposable  $T_q(\mu)$  for the quantum group  $U_q$  associated to  $W$  where  $q$  is a  $p$ th root of unity.

In the case of the quantum general linear group, Ringel duality [Erd94] translates this into a result about decomposition numbers for the Hecke algebra of type  $A$ , where now  $\mu$  labels a simple module and  $\lambda$  a Specht module. Further, Rouquier has conjectured that the coefficient of  $v^t$  occurring in  $n_{\lambda\mu}$  should correspond to the multiplicity of the simple  $D^\mu$  in the  $t$ th layer of the Jantzen filtration of the Specht module  $S^\lambda$ .

In this spirit, we can compare our results in this section with the polynomials in Figure 9 for  $n \leq 12$ . We see that in each case, the value of  $n_{\lambda,\mu}(1)$  from Figure 9 is exactly the multiplicity of  $L_n(\mu)$  in  $\Delta_n(\lambda)$ , and that there is a filtration of  $\Delta_n(\lambda)$  corresponding to the powers of  $v$  occurring in the polynomials for the  $L_n(\mu)$ s. This, together with the other Lie-like phenomena we have observed leads us to ask

**Question 9.1.** (i) For the Brauer algebra with Kazhdan-Lusztig polynomials as defined in Section 9, is it true for weights in an alcove that

$$[\Delta_n(\lambda) : L_n(\mu)] = n_{\lambda,\mu}(1) ?$$

(ii) Is there a (Jantzen?) filtration of  $\Delta_n(\lambda)$  such that the multiplicity of a simple  $L_n(\mu)$  in the  $t$ th layer is given by the coefficient of  $v^t$  in  $n_{\lambda,\mu}$ ?

As we have noted, the results in this section answer both parts in the affirmative when  $n \leq 12$  and  $\delta = 1$ .

## REFERENCES

- [AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a  $p$ -th root of unity and of semisimple groups in characteristic  $p$ : independence of  $p$* , Astérisque **220** (1994).
- [ASS06] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras I*, LMS student texts, vol. 65, Cambridge, 2006.
- [Ben91] D. J. Benson, *Representations and cohomology I*, Cambridge studies in advanced mathematics, vol. 30, CUP, 1991.
- [Bou68] N. Bourbaki, *Groupes et algèbres de Lie (Chapitres 4–6)*, Hermann, 1968.
- [Bra37] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. **38** (1937), 857–872.
- [Bro55] W. Brown, *An algebra related to the orthogonal group*, Michigan Math. J. **3** (1955), 1–22.
- [CDM05] A. G. Cox, M. De Visscher, and P. P. Martin, *The blocks of the Brauer algebra in characteristic zero*, preprint, 2005.
- [CDM06] ———, *A geometric characterisation of the blocks of the Brauer algebra*, preprint, 2006.
- [CGM03] A. G. Cox, J. J. Graham, and P. P. Martin, *The blob algebra in positive characteristic*, J. Algebra **266** (2003), 584–635.
- [CMPX06] A. G. Cox, P. P. Martin, A. E. Parker, and C. Xi, *Representation theory of towers of recollement: theory, notes, and examples*, J. Algebra **302** (2006), 340–360.

- [Deo87] V. V. Deodhar, *On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials*, J. Algebra **111** (1987), 483–506.
- [Don98] S. Donkin, *The  $q$ -Schur algebra*, LMS Lecture Notes Series, vol. 253, Cambridge University Press, 1998.
- [DT08] S. Donkin and R. Tange, *The Brauer algebra and the symplectic Schur algebra*, preprint, 2008.
- [DWH99] W. F. Doran, D. B. Wales, and P. J. Hanlon, *On the semisimplicity of the Brauer centralizer algebras*, J. Algebra **211** (1999), 647–685.
- [Erd94] K. Erdmann, *Symmetric groups and quasi-hereditary algebras*, Finite dimensional algebras and related topics (V. Dlab and L. L. Scott, eds.), Kluwer, 1994, pp. 123–161.
- [GW98] R. Goodman and N. R. Wallach, *Representations and invariants of the classical groups*, CUP, 1998.
- [Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge studies in advanced mathematics, vol. 29, CUP, 1990.
- [Jan03] J. C. Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, AMS, 2003.
- [KX01] S. König and C. Xi, *A characteristic free approach to Brauer algebras*, Trans. AMS **353** (2001), 1489–1505.
- [MW98] P. P. Martin and D. Woodcock, *The Partition algebra and a new deformation of the Schur algebras*, J. Algebra **203** (1998), 91–124.
- [MW03] ———, *Generalized blob algebras and alcove geometry*, LMS J. of Comp. and Math. **6** (2003), 249–296.
- [Rob61] G. de B. Robinson, *Representation theory of the symmetric group*, Edinburgh University Press, 1961.
- [Soe97a] W. Soergel, *Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren*, Representation Theory **1** (1997), 115–132.
- [Soe97b] ———, *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Representation Theory **1** (1997), 83–114.

*E-mail address:* A.G.Cox@city.ac.uk, M.Devisscher@city.ac.uk

CENTRE FOR MATHEMATICAL SCIENCE, CITY UNIVERSITY, NORTHAMPTON SQUARE, LONDON, EC1V 0HB, ENGLAND.

*E-mail address:* ppmartin@maths.leeds.ac.uk

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS, LS2 9JT, ENGLAND.